Code-Based Cryptography

1. Error-Correcting Codes and Cryptography
2. McEliece Cryptosystem
3. Message Attacks (ISD)
4. **Key Attacks**
5. Other Cryptographic Constructions Relying on Coding Theory

4. Key Attacks

1. Introduction
2. Support Splitting Algorithm
3. Distinguisher for GRS codes
4. Attack against subcodes of GRS codes
5. Error-Correcting Pairs
6. Attack against GRS codes
7. Attack against Reed-Muller codes
8. Attack against Algebraic Geometry codes
9. **Goppa codes still resist**

Distinguisher for Goppa codes

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Goppa Code Distinguishing (GCD) problem Difficult Problem

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IEEE Trans. Inf. Theory. 59(10), pp. 6830-6844, 2013.

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2. **General case:** best-known attacks are based on the *support splitting algorithm* and have **exponential runtime**.

   P. Loidreau, N. Sendrier
   Weak keys in McEliece public-key cryptosystem.

Distinguisher - Square Code - GRS codes

1. If $\mathcal{C}$ is a **random** linear code of length n, with high probability:

$$K(\mathcal{C}^2) = \min\left\{\binom{K(\mathcal{C})+1}{2}, n\right\}$$

2. If $\mathcal{C}$ is a **GRS** code

$$K(\mathcal{C}^2) = \min\{2K(\mathcal{C}) - 1, n\}$$

I. Márquez-Corbella, E. Martínez-Moro and R. Pellikaan.
The non-gap sequence of a subcode of a generalized Reed-Solomon code.
Designs, Codes and Cryptography, volume 66, Issue 1-3, 317-333, 2013.

C. Wieschebrink.
Cryptanalysis of the Niederreiter Public Key Scheme Based on GRS Subcodes.
PQCrypto 2010, LNCS, volume 6061, 61-72, 2010.

Distinguisher - Square Code - Alternant codes

Proposition:

- $\mathbf{a} \in \mathbb{F}_{q^m}^n$ with $a_i \neq a_j$ for all $i \neq j$
- $\mathbf{b}_1$ and $\mathbf{b}_2$ n-tuples of nonzero elements of $\mathbb{F}_{q^m}$

Then, there exists $\mathbf{b}_3 \in \mathbb{F}_{q^m}^n$ such that:

$$\mathrm{Alt}_r(\mathbf{a}, \mathbf{b}_1) * \mathrm{Alt}_s(\mathbf{a}, \mathbf{b}_2) \subseteq \mathrm{Alt}_{r+s-n+1}(\mathbf{a}, \mathbf{b}_3)$$

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Proof: Recall that $\mathrm{Alt}_r(\mathbf{a}, \mathbf{b}) \subseteq \mathrm{GRS}_r(\mathbf{a}, \mathbf{b}) = \mathrm{GRS}_{n-k}(\mathbf{a}, \mathbf{b}^{\perp})$

Let:

$$\mathbf{c}_1 \in \mathrm{Alt}_r(\mathbf{a}, \mathbf{b}_1) \implies \exists f \in \mathbb{F}_q[X]_{<n-s} \text{ such that } \mathbf{c}_1 = \mathbf{b}_1^{\perp} * f(\mathbf{a})$$

$$\mathbf{c}_2 \in \mathrm{Alt}_r(\mathbf{a}, \mathbf{b}_2) \implies \exists g \in \mathbb{F}_q[X]_{<n-r} \text{ such that } \mathbf{c}_2 = \mathbf{b}_2^{\perp} * g(\mathbf{a})$$

$$\mathbf{c}_1 * \mathbf{c}_2 = \mathbf{b}_1^{\perp}\mathbf{b}_2^{\perp} * (fg)(\mathbf{a}) \text{ with } \deg(fg) < 2n - (s+r) - 1$$

Thus $\mathbf{c}_1 * \mathbf{c}_2 \in \mathrm{GRS}_{2n-(s+r)-1}(\mathbf{a}, \mathbf{b}_3^{\perp}) \cap \mathbb{F}_q^n = \mathrm{Alt}_{s+r-n+1}(\mathbf{a}, \mathbf{b}_3^{\perp})$

Distinguisher - Square Code - Alternant codes

Thus, $\boxed{(\mathrm{Alt}_r(\mathbf{a}, \mathbf{b}))^{(2)} \subseteq \mathrm{GRS}_{2(n-r)-1}(\mathbf{a}, \mathbf{b}^{\perp})}$

To distinguish we need:

$$2(n-r) < n \Longrightarrow r > \frac{n}{2}$$

However recall that

$$\dim\left(\mathrm{Alt}_r(\mathbf{a}, \mathbf{b})\right) = n - rm \geq 0 \Longrightarrow r < \frac{n}{m} \leq \frac{n}{2} \text{ for all } m \geq 1$$

Distinguisher for Wild Goppa codes for $m = 2$

The square code of a shortened **wild Goppa code** of extension degree 2 has a **abnormal dimension.**

A. Couvreur, A. Otmani and J.P. Tillich
Polynomial Time Attack on Wild McEliece Over Quadratic Extensions.
EUROCRYPT 2014, 17—39.

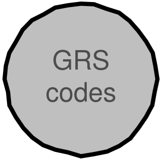
Recent results against Wild Goppa codes

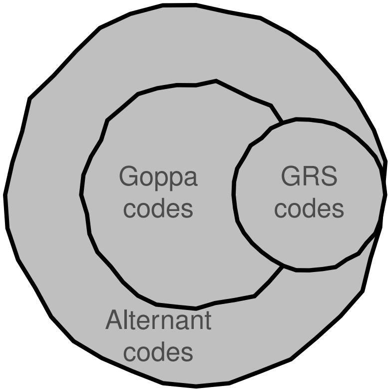
1. **Wild Goppa code** with $m = 2$

   A. Couvreur, A. Otmani and J.P. Tillich
   Polynomial Time Attack on Wild McEliece Over Quadratic Extensions.
   EUROCRYPT 2014, 17—39.

2. **Some special cases of Wild McEliece Incognito.**

   J.C. Faugère, L. Perret and F. Portzamparc
   Algebraic Attack against Variants of McEliece with Goppa Polynomial of a Special Form.
   Asiacrypt 2014, LNCS, vol 8873, 21-41. 2014.


GRS
codes


Goppa
codes
GRS
codes
Alternant
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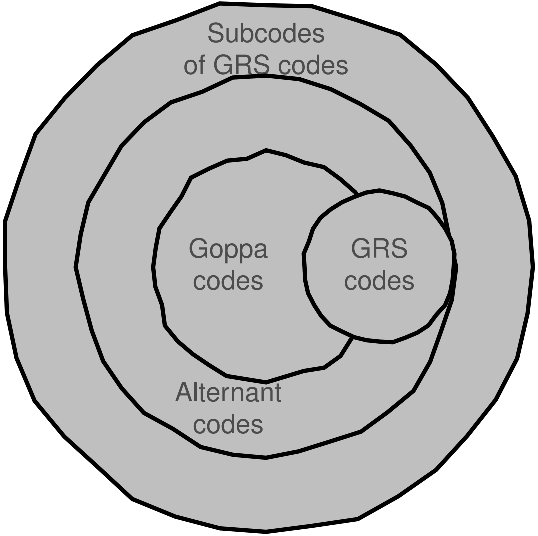
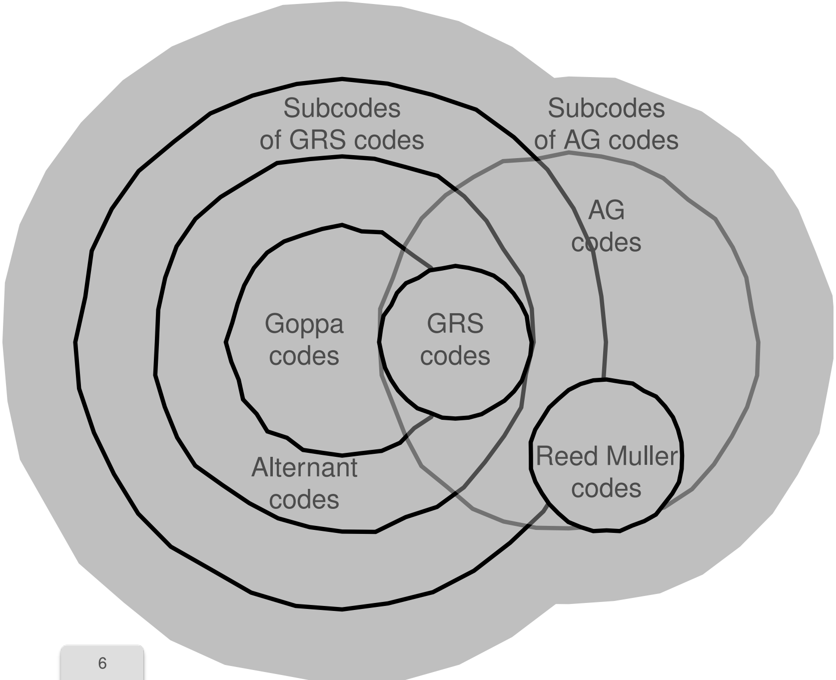

Subcodes
of GRS codes
Goppa
codes
GRS
codes
Alternant
codes


Subcodes
of GRS codes
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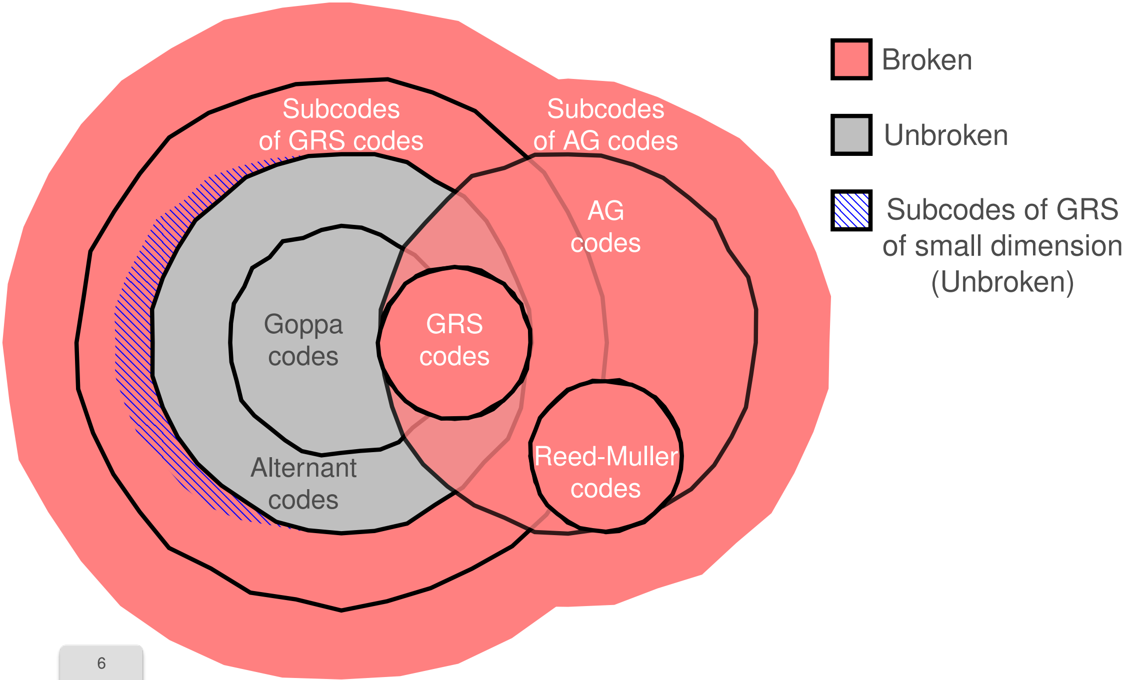
Subcodes
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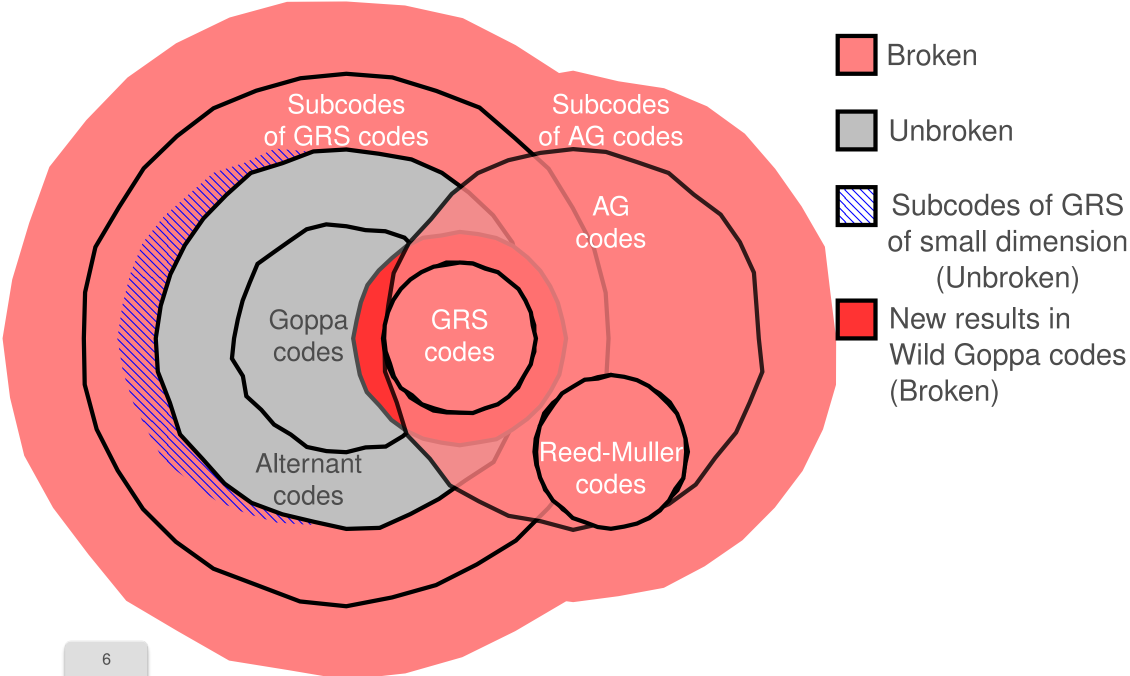
Broken
Unbroken
Subcodes of GRS
of small dimension
(Unbroken)


Subcodes of GRS codes
Subcodes of AG codes
AG codes
Goppa codes
GRS codes
Alternant codes
Reed-Muller codes
Broken
Unbroken
Subcodes of GRS of small dimension (Unbroken)
New results in Wild Goppa codes (Broken)

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