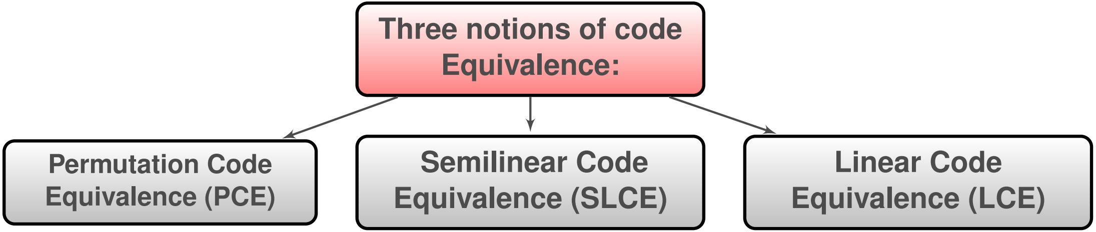
Code-Based Cryptography

1. Error-Correcting Codes and Cryptography
2. McEliece Cryptosystem
3. Message Attacks (ISD)
4. **Key Attacks**
5. Other Cryptographic Constructions Relying on Coding Theory

4. Key Attacks

The Code Equivalence Problem of Linear Codes



The Code Equivalence Problem of Linear Codes

Three notions of code Equivalence:

↓

Semilinear Code Equivalence (SLCE)

Semilinear Code Equivalence (SLE)

$$\mathcal{C}_1 \overset{\text{SLE}}{\sim} \mathcal{C}_2 \iff \exists \phi : \mathcal{C}_2 = \phi(\mathcal{C}_1)$$

$$\text{with } \phi = (\underbrace{\sigma \in S_n}_{\text{Permutation}}, \underbrace{\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{F}_q^*)^n}_{\textit{Scalar}}, \underbrace{\alpha \in \mathrm{Aut}(\mathbb{F}_q)}_{\textit{Automorphism}})$$

The Code Equivalence Problem of Linear Codes

Three notions of code Equivalence:

Permutation Code Equivalence (PCE)

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The Code Equivalence Problem of Linear Codes

Three notions of code Equivalence:

- Permutation Code Equivalence (PCE)
- Semilinear Code Equivalence (SLCE)
- Linear Code Equivalence (LCE)

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INPUT: Two $[n, k]_q$ linear codes: $\mathcal{C}_1$ and $\mathcal{C}_2$

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(Decision): Are $\mathcal{C}_1 \sim \mathcal{C}_2$?

(Computational): If $\mathcal{C}_1 \sim \mathcal{C}_2$. Find $\sigma \in S_n$ such that $\mathcal{C}_2 = \sigma(\mathcal{C}_1)$

The Code Equivalence Problem of Linear Codes

➜ **Complexity:** The **PE** problem is not NP-Complete but it is at least as hard as **Graph Isomorphism Problem**

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  - **Computation of canonical forms** for LC over $\mathbb{F}_q$, with q small.

    T. Feulner,
    The automorphism groups of linear codes and canonical representatives of their semilinear isometry classes,
    AMC, vol. 3 (4), p. 363-383, 2009

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Recall that $\mathcal{W}_{\mathcal{C}}(X) = \sum_{i=0}^{n} A_i X^i$ with $A_i = |\{\mathbf{c} \in \mathcal{C} \mid \mathrm{w}_H(\mathbf{c}) = i\}|$

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$\mathcal{W}_{\mathcal{C}_1}(X) = \mathcal{W}_{\mathcal{C}_2}(X)$ but $\mathcal{C}_1 \not\sim \mathcal{C}_2$

Consider two binary $[6,3]$ codes $\mathcal{C}_1$ and $\mathcal{C}_2$ with respective generator matrices:

$$G_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

➜ Both codes have the same weight distribution: 1, 0, 3, 0, 3, 0, 1

➜ But they are not permutation-equivalent!

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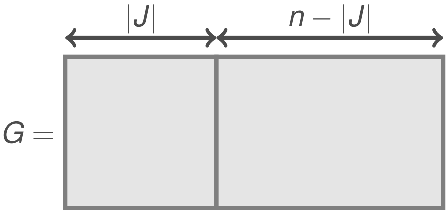

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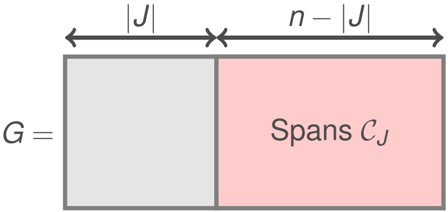

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Where $\mathcal{C}_i$ is the punctured code $\mathcal{C}$ on i

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An Example of Fully Discriminant Signature

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Thus $\sigma(1) = 3$ $\sigma(2) = 1$ $\sigma(3) = 4$ and $\sigma(4) = 2$

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Let $\mathcal{S}$ be a signature. Let J be a subset of $\{1, \ldots, n\}$ If $\mathcal{C} \sim \hat{\mathcal{C}} \implies \mathcal{C}_J \sim \hat{\mathcal{C}}_J$
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Thus: positions $\{2, 3\}$ in $\mathcal{C}$ and $\{1, 5\}$ in $\hat{\mathcal{C}}$ **cannot be discriminated**

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$$\begin{array}{lll} \mathcal{W}_{\mathcal{C}_{\{1,2\}}} = \mathcal{W}_{\hat{\mathcal{C}}_{\{2,5\}}} & \implies \sigma(\{1,2\}) = \{2,5\} \\ \mathcal{W}_{\mathcal{C}_{\{1,3\}}} = \mathcal{W}_{\hat{\mathcal{C}}_{\{2,1\}}} & \implies \sigma(\{1,3\}) = \{2,1\} \end{array}$$

Thus $\sigma(2) = 5$ and $\sigma(3) = 1$

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Thus: positions $\{2, 3\}$ in $\mathcal{C}$ and $\{1, 5\}$ in $\hat{\mathcal{C}}$ **cannot be discriminated**

$$\begin{array}{lll} \mathcal{W}_{\mathcal{C}_{\{1,2\}}} = \mathcal{W}_{\hat{\mathcal{C}}_{\{2,5\}}} & \implies \sigma(\{1, 2\}) = \{2, 5\} \\ \mathcal{W}_{\mathcal{C}_{\{1,3\}}} = \mathcal{W}_{\hat{\mathcal{C}}_{\{2,1\}}} & \implies \sigma(\{1, 3\}) = \{2, 1\} \end{array}$$

Thus $\sigma(2) = 5$ and $\sigma(3) = 1$

Some notation

From now on, let $\mathcal{C}$ be a linear code of length n defined over $\mathbb{F}_q$. We denote

- Its **dimension** by $K(\mathcal{C})$.
- Its **minimum distance** by $d(\mathcal{C})$.

Support Splitting Algorithm

The Algorithm:

N. Sendrier,
Finding the permutation between equivalent linear codes: The Support Splitting Algorithm,
IEEE Trans. on Inf. Theory, vol. 46(4), 2000.

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of increasing *“discriminancy”* such that $\mathcal{S}_r$ is **fully discriminant** for $\mathcal{C}$.

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- When $h \longrightarrow 0$, Then the **Algorithm** runs in polynomial time.

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Application in Code-Based Cryptography

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Goppa code

Let:

- ➜ $L = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_{2^m}$ with $\alpha_i \neq \alpha_j$ for all $i \neq j$.
- ➜ $g(X) \in \mathbb{F}_{2^m}[X]$ monic separable polynomial with $\deg(g) = t$ and $g(\alpha_i) \neq 0 \forall i$

$$\Gamma(L, g) = \mathrm{Alt}_t(\mathbf{a}, \mathbf{b}) = (\mathrm{GRS}_t(\mathbf{a}, \mathbf{b})) \cap \mathbb{F}_q$$

with $\mathbf{a} = L$ and $b_i = \dfrac{1}{g(a_i)}$

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➜ A Goppa code $\mathcal{C} = \Gamma(L, g)$ has :

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Is necessary to use a **large** family of codes
to make this attack ineffective

4. Key Attacks

1. Introduction
2. Support Splitting Algorithm
3. **Distinguisher for GRS codes**
4. Attack against subcodes of GRS codes
5. Error-Correcting Pairs
6. Attack against GRS codes
7. Attack against Reed-Muller codes
8. Attack against Algebraic Geometry codes
9. Goppa codes still resist