

Code-Based Cryptography

1. Error-Correcting Codes and Cryptography
2. McEliece Cryptosystem
3. Message Attacks (ISD)
4. **Key Attacks**
5. Other Cryptographic Constructions Relying on Coding Theory

4. Key Attacks

1. Introduction
2. Support Splitting Algorithm
3. Distinguisher for GRS codes
4. Attack against subcodes of GRS codes
5. Error-Correcting Pairs
6. Attack against GRS codes
7. **Attack against Reed-Muller codes**
8. Attack against Algebraic Geometry codes
9. Goppa codes still resist

Reed-Muller codes

- Were introduced by **D. Muller** in **1954**.
- **I. Reed** provided an efficient decoding algorithm.



D. E. Muller.

Application of boolean algebra to switching circuit desing and to error detection.

IEEE Trans. on Computers, 3:6–12, 1954.



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Reed-Solomon codes

Univariate polynomials

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Multivariate polynomials

Binary Reed-Muller codes

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$$\mathcal{R}(r, m) = \{f(\alpha)_{|\alpha \in \mathbb{F}_2^n} \mid f \in \mathbb{F}_2[X_1, \dots, X_m] \text{ and } \deg(f) \leq r\}$$

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Total degree bound

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Parameters of $\mathcal{R}(r, m)$

Block Length: $n = 2^m$

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in $\mathbb{F}_2[X_1, \dots, X_n]$
of degree $\leq r$ ” : $k = \sum_{i=0}^r \binom{m}{i}$

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Total degree bound

Number of Variables

Parameters of $\mathcal{R}(r, m)$

Block Length: $n = 2^m$

Minimum Distance: $d = 2^{m-r}$

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Binary Reed Muller codes - Example

Consider the code $\mathcal{R}(1, 3)$

This code is an $[8, 4, 4]_2$ code .

The monomials in $\mathbb{F}_2[X_1, X_2, X_3]$ up to degree 1 are: $\{\mathbf{1}, X_1, X_2, X_3\}$

The vectors in \mathbb{F}_2^8 associated to these monomials are:

$$\begin{array}{ll} \mathbf{1} & \longrightarrow (11111111) \\ X_1 & \longrightarrow (01010101) \\ X_2 & \longrightarrow (00110011) \\ X_3 & \longrightarrow (00001111) \end{array}$$

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**Generator matrix
for $\mathcal{R}(1, 3)$**

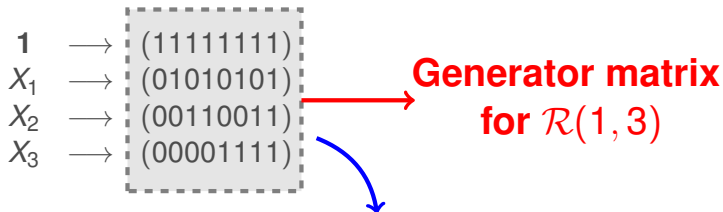
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The code $\mathcal{R}(1, m)$ has a particular shape
(After removing the first row)

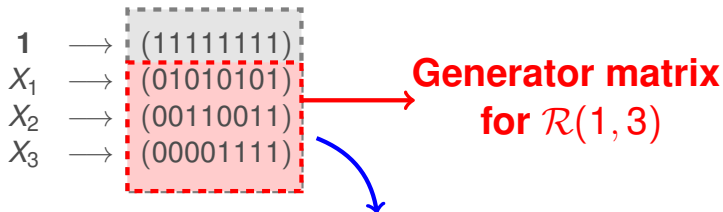
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The code $\mathcal{R}(1, m)$ has a particular shape

(After removing the first row)

The i -th column is the $(i - 1)_2$ number

(read as a binary number)

Binary Reed Muller codes - Example

Consider the code $\mathcal{R}(2, 4)$ This code is an $[16, 11, 4]_2$ code .

The monomials in $\mathbb{F}_2[X_1, X_2, X_3, X_4]$ up to degree 1 are:

$$\{\mathbf{1}, X_1, X_2, X_3, X_4, X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4, X_3X_4\}$$

The vectors in \mathbb{F}_2^{16} associated to these monomials are:

$\mathbf{1}$	\longrightarrow	(11111111 11111111)
X_1	\longrightarrow	(01010101 01010101)
X_2	\longrightarrow	(00110011 00110011)
X_3	\longrightarrow	(00001111 00001111)
X_4	\longrightarrow	(00000000 11111111)
X_1X_2	\longrightarrow	(00010001 00010001)
X_1X_3	\longrightarrow	(00000101 00000101)
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**Generator matrix
for $\mathcal{R}(2, 4)$**

Properties of Reed Muller codes

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Property 2. $\mathcal{R}(m, m) = \mathbb{F}_2^n$

Property 3. $\mathcal{R}(r, m)^\perp = \mathcal{R}(m - r - 1, m)$

Proof: Take notice that:

1. $\dim(\mathcal{R}(r, m)) + \dim(\mathcal{R}(m - r - 1, m)) = 2^m$
2. $\mathcal{R}(r, m) * \mathcal{R}(m - r - 1, m) = \mathcal{R}(m - 1, m)$
3. The code $\mathcal{R}(m - 1, m)$ is the code of all even weight vectors.

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★ : $\mathcal{R}(r_1, m), \mathcal{R}(r_2, m) \xrightarrow{\mathcal{O}(n^4) \text{ bit operations}} \mathcal{R}(r_1 + r_2, m) \text{ if } r_1 + r_2 \leq m - 2$

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We need to find minimum weight codewords.

Properties of Reed Muller codes

Proposition [Chizhov - Borodin]

Let $t = a(m - 1) + br$. Then, $\mathcal{R}(r, m) \xrightarrow{\mathcal{O}(n^4) \text{ bit operations}} \mathcal{R}(t, m)$

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1. $a = 0, b > 0$

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3. $a \leq 0, b > 0$

Take notice that $m - 1 - t = \underbrace{(1 - a)}_{a' \geq 0} (m - 1) - br$. Apply **case 2** to $\mathcal{R}(m - 1 - t, m)$

Reed-Muller codes for the McEliece scheme



Reed-Muller codes



V. Sidelnikov.

A public-key cryptosystem based on Reed-Muller codes.
Discrete Math. Appl., 4(3):191–207, 1994.

Parameters	Key size	Security level
$[1024, 176, 128]_2$	22.5 ko	2^{72}
$[2048, 232, 256]_2$	59, 4 ko	2^{93}

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Attacks against this proposal:



L. Minder and A. Shokrollahi.

Cryptanalysis of the Sidelnikov cryptosystem.

In EUROCRYPT 2007, pages 347–360, 2007.



I. V. Chizhov, and M. A. Borodin.

The failure of McEliece PKC based on Reed-Muller codes.

IACR Cryptology ePrint Archive, 287, 2013.

Attack

Public Key: $\mathcal{K}_{\text{pub}} = \left\{ \begin{array}{l} \text{a gen. matrix of } \mathcal{C} = \mathcal{R}^\sigma(r, m) \text{ for some permutation } \sigma \\ \text{and } t = \left\lfloor \frac{d(\mathcal{C})-1}{2} \right\rfloor \end{array} \right.$

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The Algorithm:

1. Compute the code $\mathcal{R}^\sigma(1, m)$ from $\mathcal{R}^\sigma(r, m)$

Proposition:

- If $\gcd(r, m-1) = 1$, i.e. $1 = a(m-1) + br$. Then,

$$\mathcal{R}(r, m) \xrightarrow{\text{Prop. Chizhov-Borodin}} \mathcal{R}(1, m)$$

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Proposition:

- If $\gcd(r, m-1) = d$. Then,

$$\begin{array}{ccccc} \mathcal{R}(r, m) & \xrightarrow{\text{Prop. Chizhov-Borodin}} & \mathcal{R}(d, m) & \xrightarrow{\perp} & \mathcal{R}(m-d-1, m) \\ & & \downarrow \oplus & & \\ & & \mathcal{R}(d-1, m) & & \end{array}$$

$$\mathcal{R}(d-1, m), \mathcal{R}(m-d-1, m) \xrightarrow{*} \mathcal{R}(m-2, m) = \mathcal{R}(1, m)$$

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