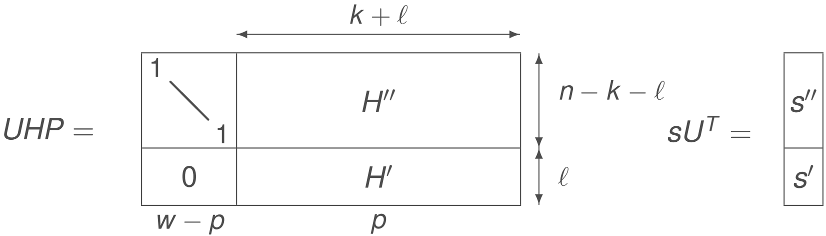
3. Message Attack (ISD)

1. From Generic Decoding to Syndrome Decoding
2. Combinatorial Solutions: Exhaustive Search and Birthday Decoding
3. Information Set Decoding: the Power of Linear Algebra
4. Complexity Analysis
5. Lee and Brickell Algorithm
6. **Stern/Dumer Algorithm**
7. May, Meurer, and Thomae Algorithm
8. Becker, Joux, May, and Meurer Algorithm
9. Generalized Birthday Algorithm for Decoding
10. Decoding One Out of Many

Stern Algorithm – Dumer Algorithm

Idea: combine Lee & Brickell algorithm and birthday decoding



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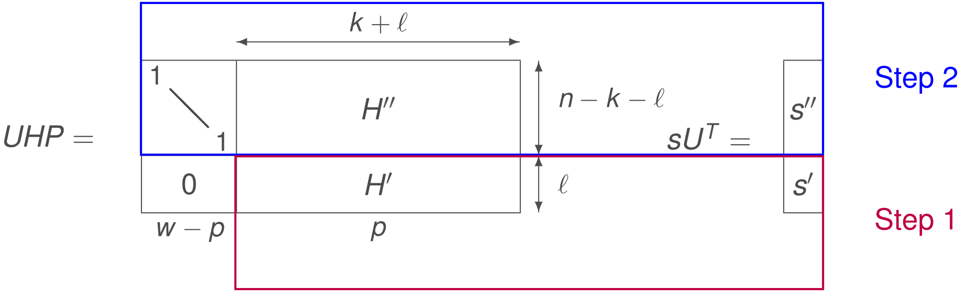

Step 1: Find all $e' \in \mathrm{CSD}(H', s', p)$
Step 2: Check $\mathrm{wt}(e' H''^T + s'') = w - p$

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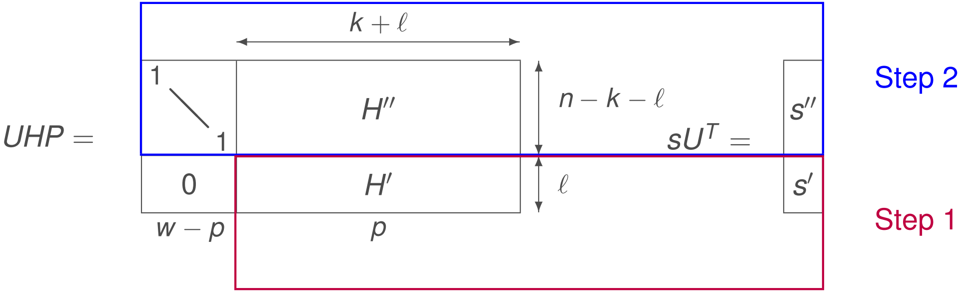

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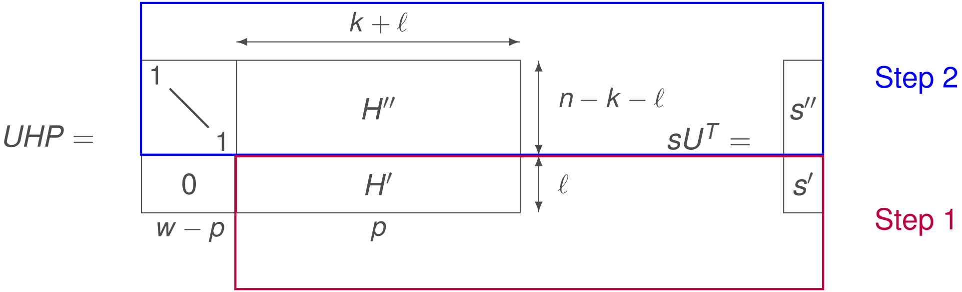

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If step 1 is solved by birthday decoding $\rightarrow$ Dumer Algorithm

Dumer Algorithm

input: $H \in \{0,1\}^{(n-k)\times n}$, $s \in \{0,1\}^{n-k}$, integer $w > 0$, two parameters p and ℓ

output: $e \in \{0,1\}^n$ such that $eH^T = s$ and $\mathrm{wt}(e) = w$

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repeat:
    pick a permutation matrix P
    compute U, H', H'', s', s''

$$UHP = \left(\begin{array}{c|c} 1 \ddots 1 & H'' \\ \hline 0 & H' \end{array}\right) \updownarrow \ell$$

$$sU^T = \left(\begin{array}{c} s'' \\ \hline s' \end{array}\right)$$

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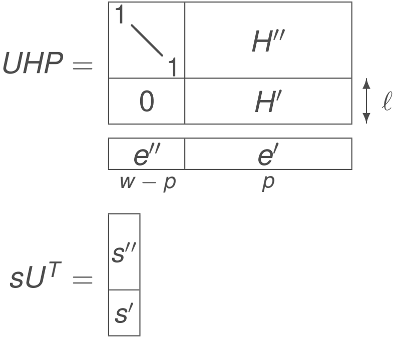
repeat:
    pick a permutation matrix P
    compute U, H', H'', s', s''
    solve $\mathrm{CSD}(H', s', p)$ *(birthday decoding)*
    for all $e' \in \mathrm{CSD}(H', s', p)$
        $e'' \leftarrow e' H''^T + s''$
        if $\mathrm{wt}(e'') = w - p$
            return $(e'', e')P$



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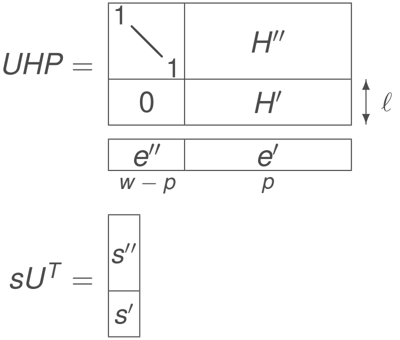

Note: Stern's algorithm (1989) was the first to use birthday decoding, Dumer's algorithm (1991) is only marginally better

We will refer now to the Stern/Dumer Algorithm

Stern/Dumer Algorithm – Complexity Analysis (1/2)

Iteration cost: $\mathcal{K} = n(n-k-\ell) + 2\sqrt{\binom{k+\ell}{p}} + \frac{\binom{k+\ell}{p}}{2^\ell} + \frac{\binom{k+\ell}{p}}{2^\ell}$

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Gaussian elimination

Birthday decoding

Final check

Stern/Dumer Algorithm – Complexity Analysis (1/2)

In general, we can write

$$\mathcal{K} = K_0 \cdot n(n-k-\ell) + K_1 \cdot \sqrt{\binom{k+\ell}{p}} + K_2 \cdot \frac{\binom{k+\ell}{p}}{2^\ell}$$

where K_0, K_1, and K_2 are small (implementation dependent) constants

we will set $K_0 = K_1 = K_2 = 1$ to simplify the formula

Stern/Dumer Algorithm – Complexity Analysis (1/2)

We will simply write $\mathcal{K} = n(n-k-\ell) + \sqrt{\binom{k+\ell}{p}} + \frac{\binom{k+\ell}{p}}{2^\ell}$ up to a constant factor

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Success probability: $\mathcal{P}_\infty = \frac{\binom{k+\ell}{p}\binom{n-k-\ell}{w-p}}{\binom{n}{w}}$ and $\mathcal{N}_\infty = \frac{1}{\mathcal{P}_\infty} = \frac{\binom{n}{w}}{\binom{k+\ell}{p}\binom{n-k-\ell}{w-p}}$

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To be minimized over p and ℓ (positive integers)

Stern/Dumer Algorithm – Complexity Analysis (2/2)

The optimization parameters p and ℓ grow with the problem parameters (n, k, w)

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In most situations, the above formula is minimal when the addends are equal

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