Code-Based Cryptography

McEliece Cryptosystem

I. Márquez-Corbella

2. McEliece Cryptosystem

1. Formal Definition
2. Security-Reduction Proof
3. McEliece Assumptions
4. Notions of Security
5. Critical Attacks - Semantic Secure Conversions
6. Reducing the Key Size
7. **Reducing the Key Size - LDPC codes**
8. Reducing the Key Size - MDPC codes
9. Implementation

Low-density parity-check (LDPC) codes

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1996: MacKay and Neal (re)-discovered LDPC codes

D. J.C. MacKay and R. M. Neal.
Near shannon limit performance of Low Density Parity Check codes.
Electronics Letters, 32:1645âĂŞ1646, 1996.

Representation for LDPC codes

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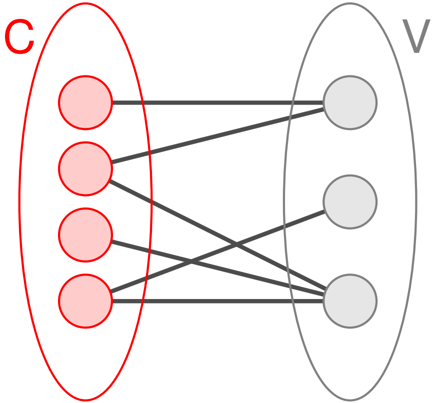

Bipartite Graph

Representation for LDPC codes

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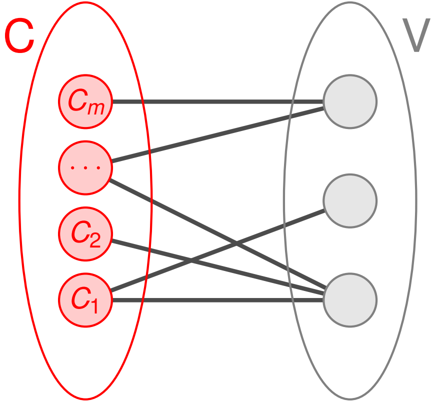

Tanner Graph

Variable nodes

$V_j \longleftrightarrow j$-th column of H

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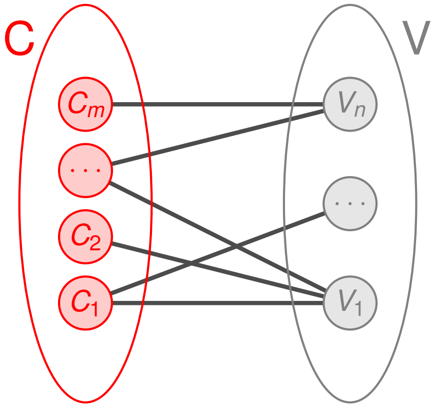

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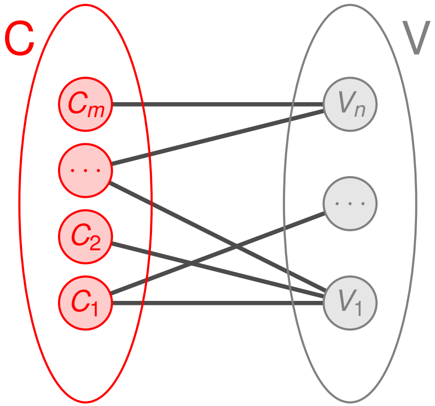

Tanner Graph

Variable nodes

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Edges

$e_{i,j} = \{C_i, V_j\} \longleftrightarrow h_{i,j} = 1$ in H

Example

Let $\mathcal{C}$ be an $[10, 7]$ binary LDPC code with parity-check matrix:

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{F}_2^{3\times 10}$$

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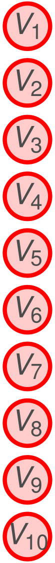

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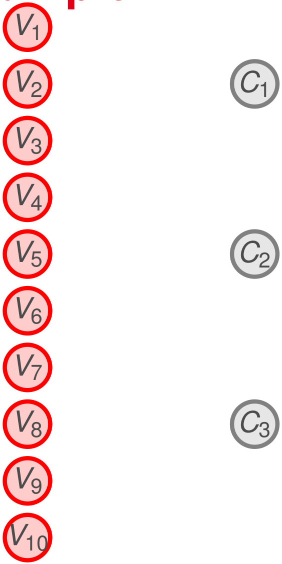

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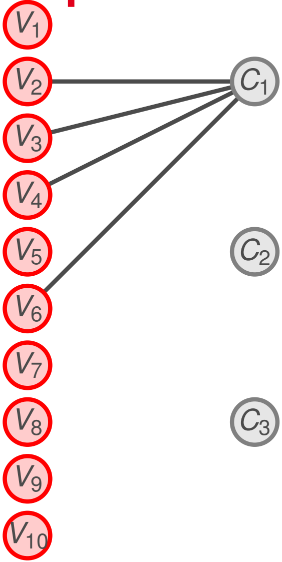

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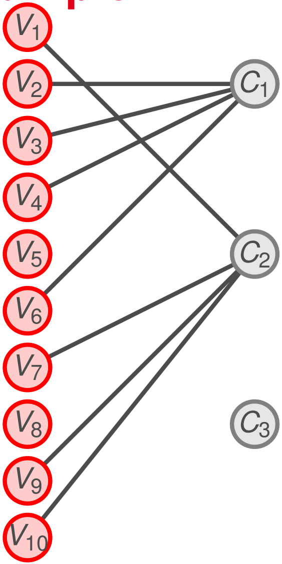

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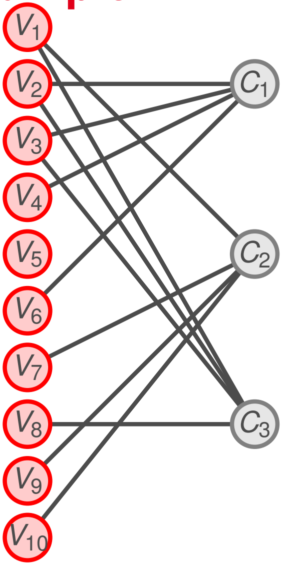

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Bit-Flipping decoding algorithm

Step 1 - Iteration I Compute:

$f_j :=$ Number of unsatisfied parity-check equations of V_j with $j = 1, \ldots, n$
$f := \max(f_1, \ldots, f_n)$

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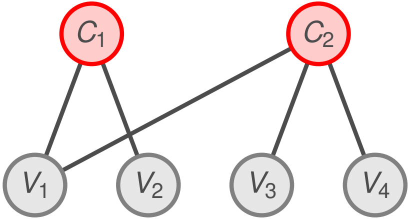
Step 3 - Iteration I **Stop Criterion**

- Success: If $f = 0$ and I $< I_{max}$
- Failure: If $f \neq 0$ and I $= I_{max}$

Bit-Flipping Decoding - Example

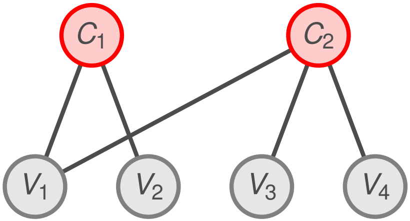
Received Data: $(0, 1, 1, 1)$



Bit-Flipping Decoding - Example

Received Data: $(0, 1, 1, 1)$

	V_1	V_2	V_3	V_4
Current	0	1	1	1
C_0	✗	✗	–	–
C_1	✓	–	✓	✓
f_j	1	1	0	0



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	V_1	V_2	V_3	V_4
Current	0	1	1	1
C_0	✗	✗	–	–
C_1	✓	–	✓	✓
f_j	1	1	0	0
Updated	1	0	1	1
C_0	✗	✗	–	–
C_1	✗	–	✗	✗
f_j	2	1	1	1

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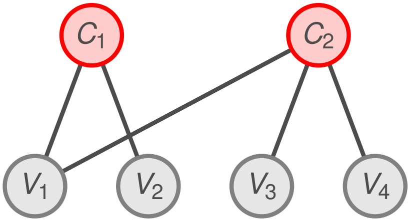

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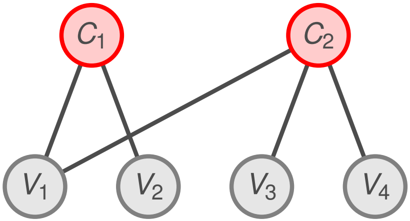

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	V_1	V_2	V_3	V_4
Current	0	1	1	1
C_0	✗	✗	–	–
C_1	✓	–	✓	✓
f_j	1	1	0	0
Updated	1	0	1	1
C_0	✗	✗	–	–
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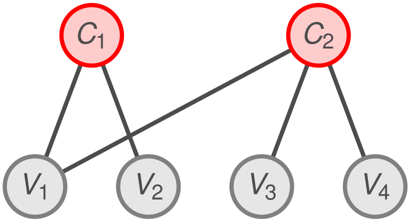

Decoding Result: (0, 0, 1, 1)

Variants based on LDPC codes

Using pure LDPC codes

C. Monico, J. Rosenthal, A. Shokrollahi.
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✗ **Weakness:** Search for low weight codewords in the dual of the public code

Variants using QC-LDPC codes

➢ First proposal

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A. Otmani, J.P. Tillich, and L. Dallot.
Cryptanalysis of two McEliece cryptosystems based on quasi-cyclic codes.
Special Issues of Mathematics in Computer Science, pp. 126-140, 2010.

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➢ **New variant:**

M. Baldi, M. Bodrato, and F. Chiaraluce.
A new analysis of the McEliece cryptosystem based on QC-LDPC codes.
In SCN 2008, pp. 246-262.

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