

Code-Based Cryptography

Error-Correcting Codes and Cryptography

1. Error-Correcting Codes and Cryptography

1. Introduction I - Cryptography
2. Introduction II - Coding Theory
3. Encoding (Linear Transformation)
4. Parity Checking
5. Error Correcting Capacity
6. **Decoding (A Difficult Problem)**
7. Reed-Solomon Codes
8. Goppa Codes
9. McEliece Cryptosystem

Decoder



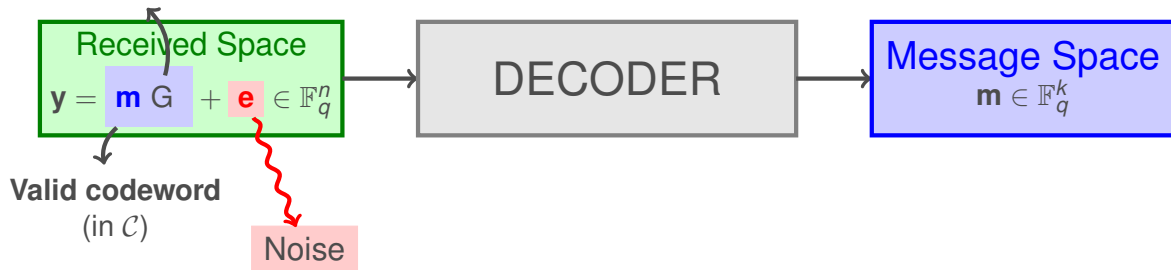
Decoder

Encoding matrix



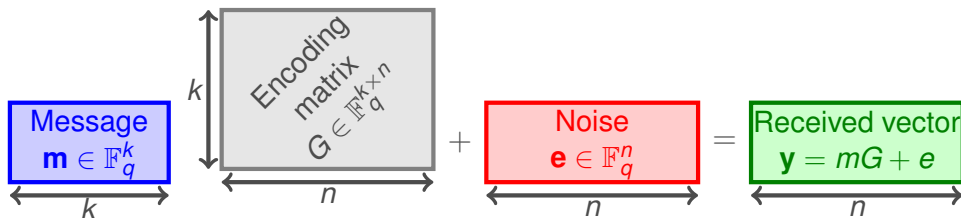
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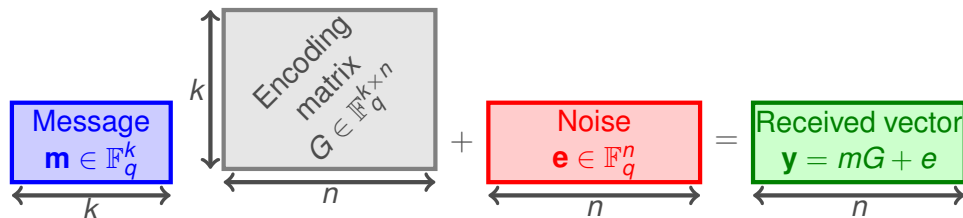


$$\text{Decoder} \left(\underbrace{\text{Encoder} (\text{Message})}_{\text{codeword}} + \text{Noise} \right) = \text{Message}$$

Minimum Distance Decoding (MDD)



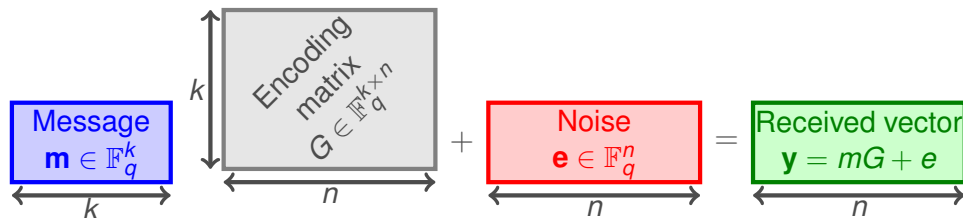
Minimum Distance Decoding (MDD)



Instances:

- A matrix $G \in \mathbb{F}_q^{k \times n}$ (generator matrix for \mathcal{C})
- A received vector $\mathbf{y} \in \mathbb{F}_q^n$

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Instances:

- A matrix $G \in \mathbb{F}_q^{k \times n}$ (generator matrix for \mathcal{C})
- A received vector $\mathbf{y} \in \mathbb{F}_q^n$

Output:

(Search - MDD): Find $\mathbf{m} \in \mathbb{F}_q^k$ to minimize

$$d_H(\mathbf{y}, \mathbf{m}G)$$

Brute Force

Let y be
the received word

First idea: Brute Force

Compute the Hamming distance of the received word with all codewords.

Brute Force

Let \mathbf{y} be
the received word

\mathbf{c}_1
\mathbf{c}_2
\vdots
\mathbf{c}_N

with $N = q^k$

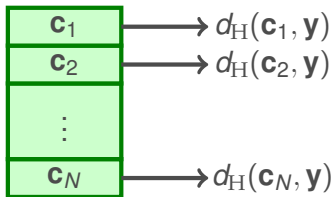
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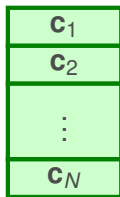
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1. Enumerate all codewords of \mathcal{C} .
2. If \mathbf{y} is the received word.

Compute the Hamming distance $d_H(\mathbf{c}, \mathbf{y})$, $\forall \mathbf{c} \in \mathcal{C}$

Brute Force

Let \mathbf{y} be
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$$\rightarrow d_H(\mathbf{c}_1, \mathbf{y})$$

$$\rightarrow d_H(\mathbf{c}_2, \mathbf{y})$$

$$\rightarrow d_H(\mathbf{c}_N, \mathbf{y})$$

with $N = q^k$

Return: \mathbf{c}_i such that
 $d_H(\mathbf{c}_i, \mathbf{y})$ is minimized

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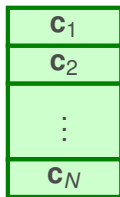
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The **complexity** is $\mathcal{O}(nq^k)$

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Syndrome

Let \mathcal{C} be an $[n, k]_q$ code with parity check matrix H

$$\mathbf{c} \in \mathcal{C} \implies H\mathbf{c}^T = \mathbf{0}$$

Syndrome of a vector

The **syndrome of a vector** $\mathbf{x} \in \mathbb{F}_q^n$ is the vector $S(\mathbf{x}) = H\mathbf{x}^T \in \mathbb{F}_q^{n-k}$

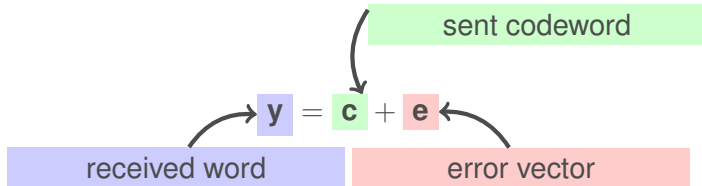
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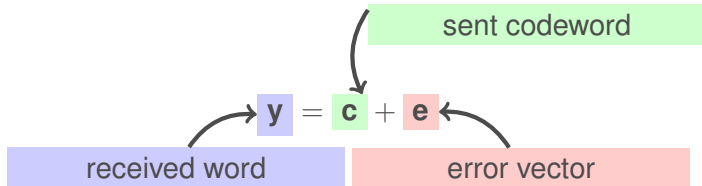
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$$H\mathbf{y}^T = H(\mathbf{c} + \mathbf{e})^T = \underbrace{H\mathbf{c}^T}_{=0} + H\mathbf{e}^T = H\mathbf{e}^T$$

Syndrome Decoding - Lookup table

Let \mathbf{y} be
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Suppose we want to correct all patterns of $\leq t$ errors

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$S(\mathbf{e}_1) = S_1$
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\vdots
$S(\mathbf{e}_N) = S_N$

Suppose we want to correct all patterns of $\leq t$ errors

1. Precompute the syndrome corresponding to $0, 1, \dots, t$
Number of Syndromes to pre-compute and store:

$$\binom{n}{0} + (q-1)\binom{n}{1} + (q-1)^2\binom{n}{2} + \dots + (q-1)^t\binom{n}{t}$$

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If $S_i = S(\mathbf{y})$, **Return:** $\mathbf{y} - \mathbf{e}_i$

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2. Compute the Syndrome of the received word $S(\mathbf{y})$

→ If there exists $\mathbf{e} \in \mathbb{F}_q^n$ with $w_H(\mathbf{e}) \leq t : S(\mathbf{e}) = S(\mathbf{y}) \implies$ **Return:** $\mathbf{y} - \mathbf{e}$

→ Otherwise, \implies **Return:** FAILURE

Gilbert-Varshamov bound

GV bound

$$\sum_{i=0}^{d-2} (q-1)^i \binom{n-1}{i} < q^{n-k} \implies \text{Exists an } [n, k, d]_q \text{ code}$$

Proof:

Gilbert-Varshamov bound

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We construct by induction the columns $h_1, \dots, h_n \in \mathbb{F}_q^{n-k}$ of H .

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- \dots
- $h_j \in \mathbb{F}_q^{n-k}$ any vector that is **NOT** a **LC** of $\leq (d-2)$ of $\{h_1, \dots, h_{j-1}\}$

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Let $j < n$. Exists h_{j+1} with the above property if:

$$\sum_{i=0}^{d-2} \binom{j}{i} (q-1)^i$$

\leq

$$q^{n-k} - 1$$

Number of **LC** of $\leq (d-2)$
of $\{h_1, \dots, h_j\}$

Total number of vectors
in \mathbb{F}_q^n but one

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Let $j < n$. Exists h_{j+1} with the above property if:

$$\sum_{i=0}^{d-2} \binom{n-1}{i} (q-1)^i \leq q^{n-k} - 1$$

Diagram illustrating the inequality:

- The left side of the inequality, $\sum_{i=0}^{d-2} \binom{n-1}{i} (q-1)^i$, is enclosed in a red box. Below it, a red dashed box contains the text: "Number of **LC** of $\leq (d-2)$ of $\{h_1, \dots, h_j\}$ ". A red arrow points from this text to the left side of the inequality.
- The right side of the inequality, $q^{n-k} - 1$, is enclosed in a gray box. Below it, a gray dashed box contains the text: "Total number of vectors in \mathbb{F}_q^n but one". A gray arrow points from this text to the right side of the inequality.

Gilbert Varshamov distance

Gilbert-Varshamov (GV) distance

The GV distance of an $[n, k]_q$ code is the **maximal integer** d_0 such that:

$$\sum_{i=0}^{d_0-1} \binom{n}{i} (q-1)^i \leq q^{n-k}$$

Number of codewords of a given weight

$$A_w(\mathcal{C}) = |\{\mathbf{c} \in \mathcal{C} \mid w_H(\mathbf{c}) = w\}|$$

↑
Distinct codewords in \mathcal{C}
of weight exactly w

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In a binary random code: $\mathbb{E}[A_w(\mathcal{C})] = \frac{\binom{n}{w} |\mathcal{C}|}{2^n} = \frac{\binom{n}{w}}{2^{n-k}}$

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In average:

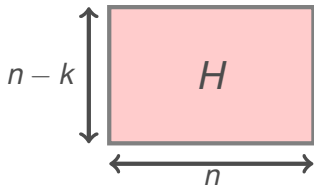
$$\begin{aligned} \text{Exists } \mathbf{c} \in \mathcal{C} \text{ with } w_H(\mathbf{c}) = w &\iff \binom{n}{w} > 2^{n-k} \\ &\iff w \text{ is closed to the GV distance} \end{aligned}$$

The Syndrome Decoding (SD) problem

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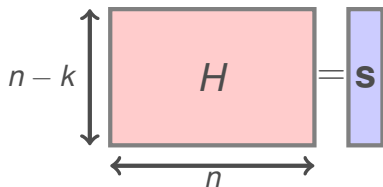


Input:

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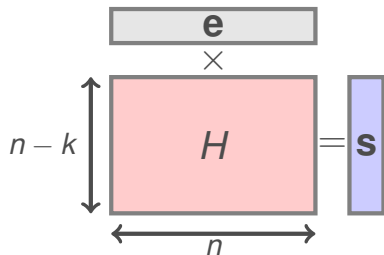


Input:

- A matrix $H \in \mathbb{F}_2^{(n-k) \times n}$
- A syndrome $s \in \mathbb{F}_2^{n-k}$

The Syndrome Decoding (SD) problem

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Input:

- A matrix $H \in \mathbb{F}_2^{(n-k) \times n}$
- A syndrome $\mathbf{s} \in \mathbb{F}_2^{n-k}$
- A weight $w \in \mathbb{Z}$

The Syndrome Decoding (SD) problem

The Syndrome Decoding (SD) problem

Output

(Decision): Does $\mathbf{e} \in \mathbb{F}_2^n$ of $w_H(\mathbf{e}) \leq w$ such that $\mathbf{e}H^T = \mathbf{s}$ exists?

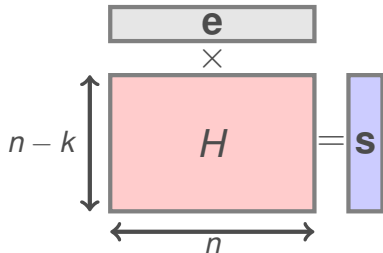
NP-complete



E. R. Berlekamp, R. J. McEliece and H. C. A. van Tilborg.
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NP-complete

(Computational): Find $\mathbf{e} \in \mathbb{F}_2^n$ of $w_H(\mathbf{e}) \leq w$ such that $\mathbf{e}H^T = \mathbf{s}$

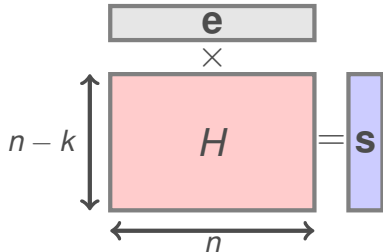
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General Decoding

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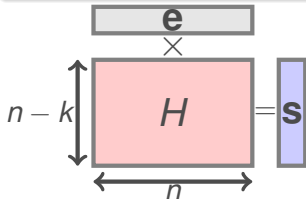
→ A parity-check matrix $H \in \mathbb{F}_2^{(n-k) \times n}$

→ A received vector $\mathbf{y} \in \mathbb{F}_2^{n-k}$

→ A weight $w \in \mathbb{Z}$

SD

Find $\mathbf{e} \in \mathbb{F}_2^n$ of $w_H(\mathbf{e}) \leq w$ such that
 $\mathbf{e}H^T = \mathbf{y}H^T = \mathbf{s}$



General Decoding

Input:

→ A parity-check matrix $H \in \mathbb{F}_2^{(n-k) \times n}$

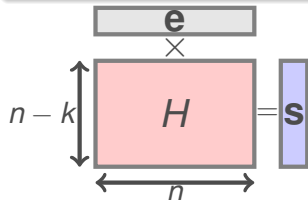
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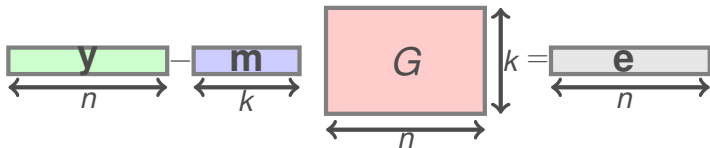
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MDD

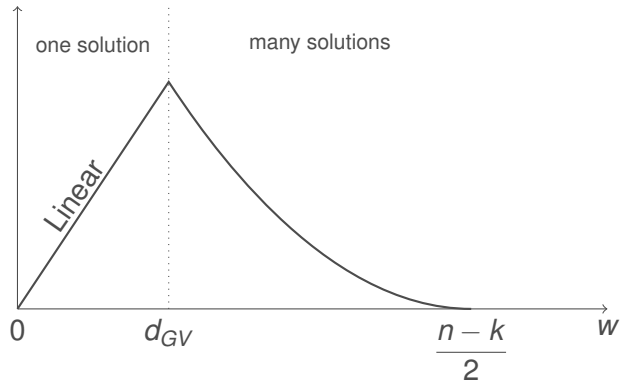
Find $\mathbf{m} \in \mathbb{F}_2^k$ such that
 $w_H(\mathbf{y} - \mathbf{m}G) \leq w$



Computational Analysis of Syndrome Decoding

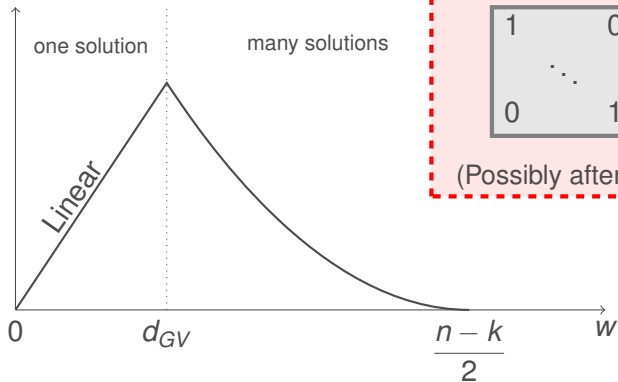
cost (log) of ISD

Binary codes



Computational Analysis of Syndrome Decoding

cost (log) of ISD
Binary codes



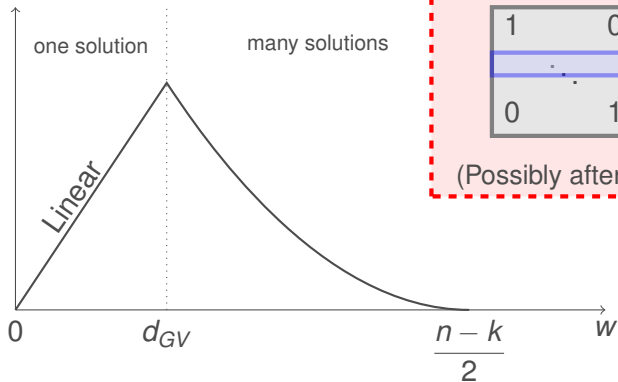
Case $w = \frac{n-k}{2}$

1	0	
	\ddots	
0	1	

(Possibly after permuting some columns)

Computational Analysis of Syndrome Decoding

cost (log) of ISD
Binary codes



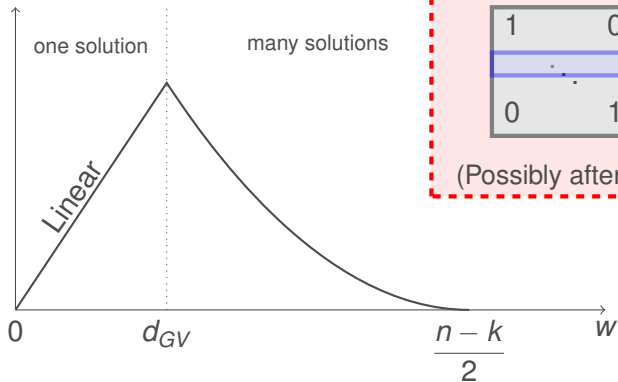
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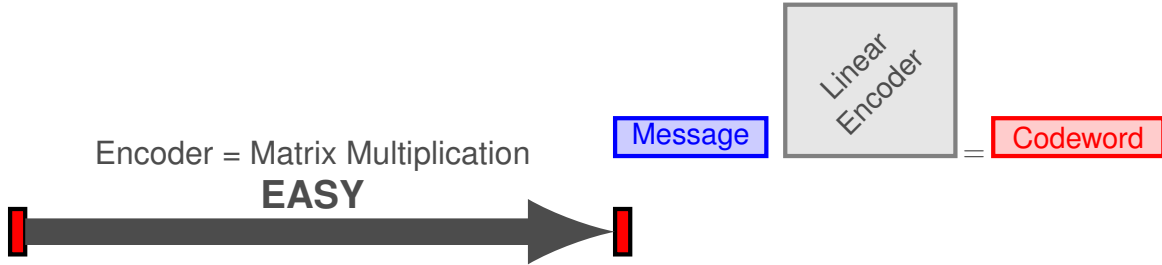
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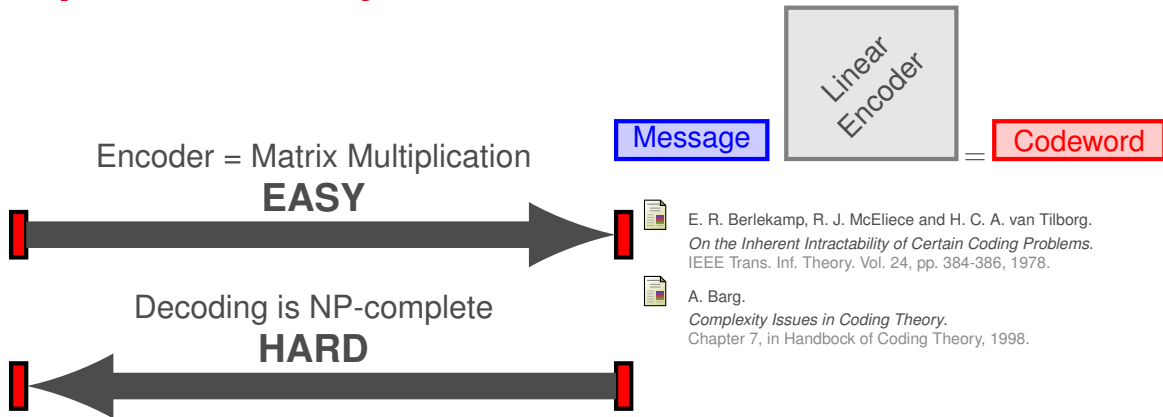
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 $\iff w$ is closed to the GV distance

Trapdoor one-way functions - Decoder



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Encoder = Matrix Multiplication

EASY

Message

Linear
Encoder

Codeword

Decoding is NP-complete

HARD

Efficient decoder for certain families of codes

EASY (with TRAPDOOR information)



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