Code-Based Cryptography

Error-Correcting Codes and Cryptography

I. Márquez-Corbella

informatiques mathématiques

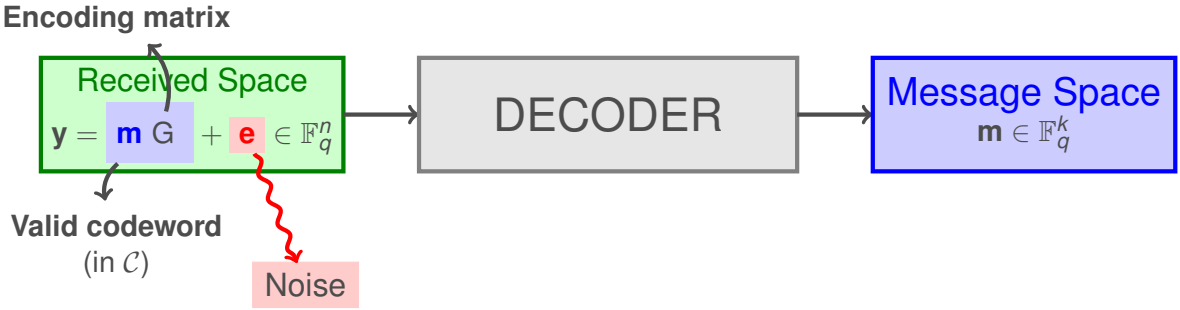
1. Error-Correcting Codes and Cryptography

1. Introduction I - Cryptography
2. Introduction II - Coding Theory
3. Encoding (Linear Transformation)
4. Parity Checking
5. Error Correcting Capacity
6. **Decoding (A Difficult Problem)**
7. Reed-Solomon Codes
8. Goppa Codes
9. McEliece Cryptosystem

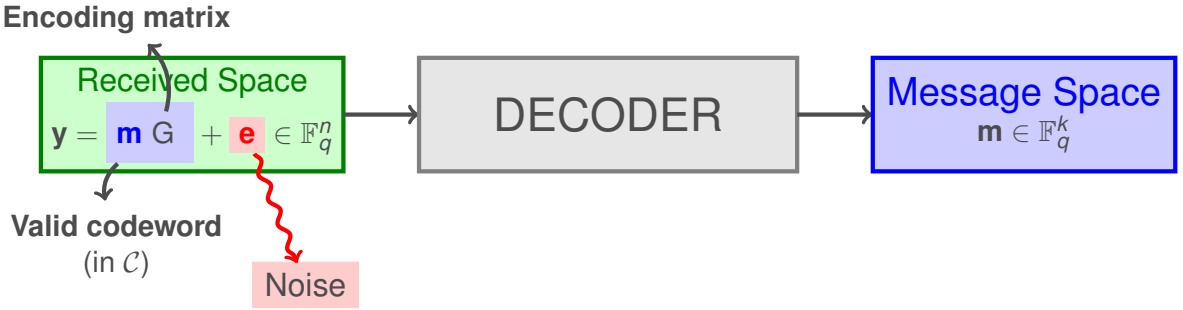
Decoder


Received Space
$\mathbf{y} = \mathbf{m}\,\mathrm{G} + \mathbf{e} \in \mathbb{F}_q^n$
DECODER
Message Space
$\mathbf{m} \in \mathbb{F}_q^k$

Decoder

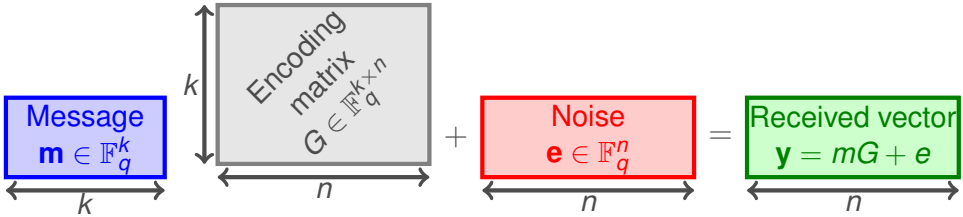

Encoding matrix
Received Space
$\mathbf{y} = \mathbf{m}\,\mathrm{G} + \mathbf{e} \in \mathbb{F}_q^n$
DECODER
Message Space
$\mathbf{m} \in \mathbb{F}_q^k$
Valid codeword
(in $\mathcal{C}$)
Noise

Decoder



$$\texttt{Decoder}\big(\underbrace{\texttt{Encoder}(\texttt{Message})}_{\text{codeword}} + \texttt{Noise}\big) = \texttt{Message}$$

Minimum Distance Decoding (MDD)



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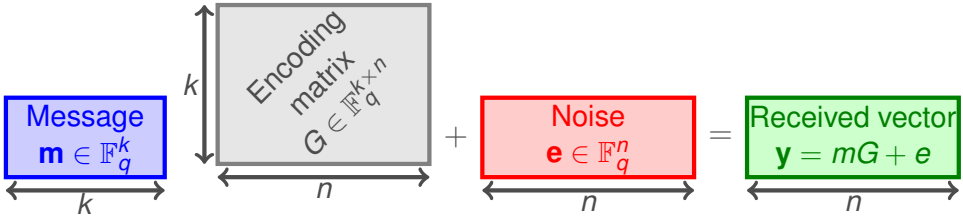

Instances:

- A matrix $G \in \mathbb{F}_q^{k \times n}$ (generator matrix for $\mathcal{C}$)
- A received vector $\mathbf{y} \in \mathbb{F}_q^n$

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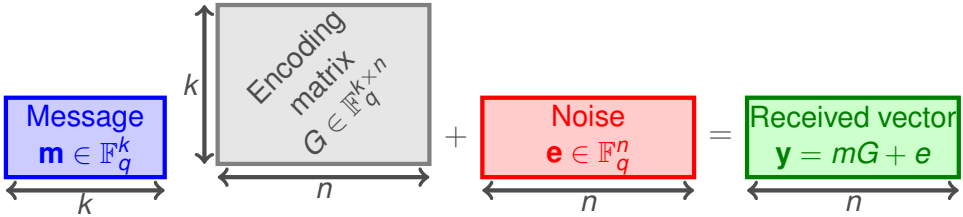

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- A matrix $G \in \mathbb{F}_q^{k \times n}$ (generator matrix for $\mathcal{C}$)
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Output:

(Search - MDD): Find $\mathbf{m} \in \mathbb{F}_q^k$ to minimize

$$d_{\mathrm{H}}\left(\mathbf{y}, \mathbf{m}G\right)$$

Brute Force

Let **y** be
the received word

First idea: Brute Force

Compute the Hamming distance of the received word with all codewords.

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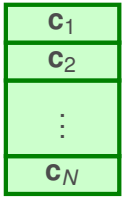

with $N = q^k$

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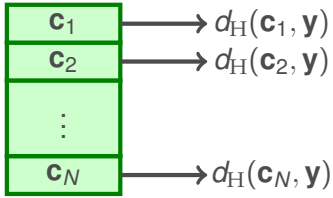

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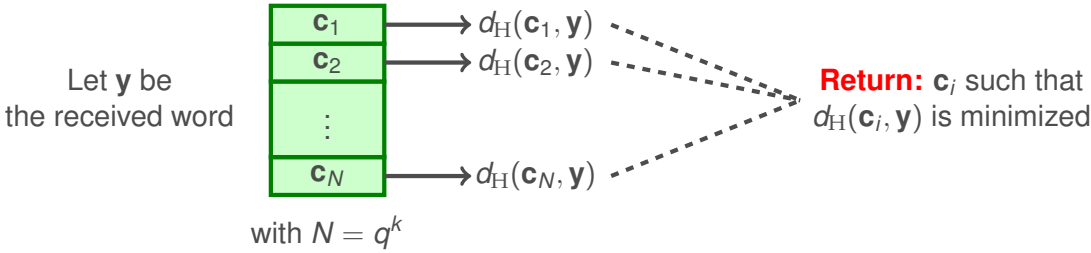

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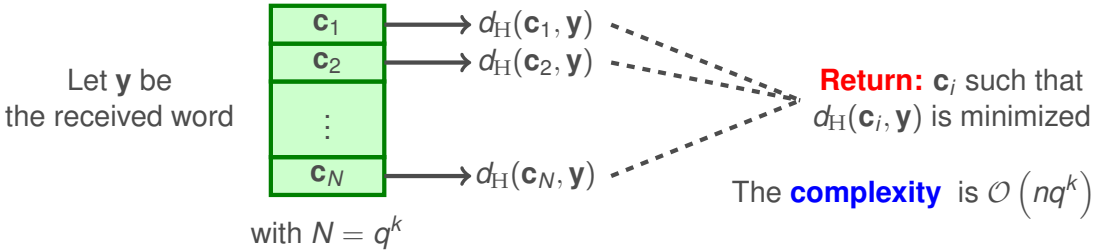

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Syndrome

Let $\mathcal{C}$ be an $[n, k]_q$ code with parity check matrix H

$$\mathbf{c} \in \mathcal{C} \implies H\mathbf{c}^T = \mathbf{0}$$

Syndrome of a vector

The **syndrome of a vector** $\mathbf{x} \in \mathbb{F}_q^n$ is the vector $\mathrm{S}(\mathbf{x}) = H\mathbf{x}^T \in \mathbb{F}_q^{n-k}$

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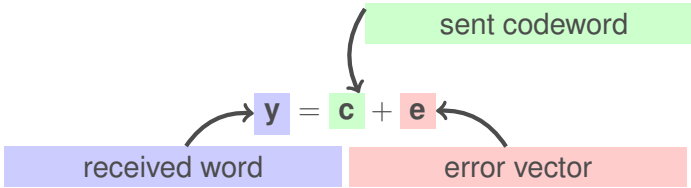

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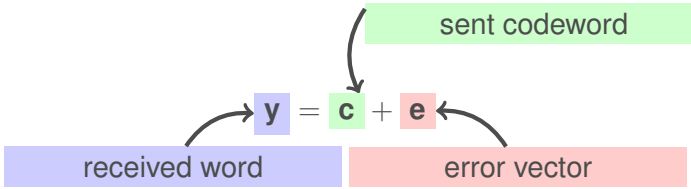

$$H\mathbf{y}^T = H(\mathbf{c} + \mathbf{e})^T = \underbrace{H\mathbf{c}^T}_{=\mathbf{0}} + H\mathbf{e}^T = H\mathbf{e}^T$$

Syndrome Decoding - Lookup table

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Suppose we want to correct all patterns of $\leq t$ errors

Syndrome Decoding - Lookup table

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$\vdots$
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Suppose we want to correct all patterns of $\leq t$ errors

1. **Precompute the syndrome corresponding to** $0, 1, \ldots, t$
   Number of Syndromes to pre-compute and store:

$$\binom{n}{0} + (q-1)\binom{n}{1} + (q-1)^2\binom{n}{2} + \ldots + (q-1)^t\binom{n}{t}$$

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If $S_i = S(\mathbf{y})$, **Return:** $\mathbf{y} - \mathbf{e}_i$

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2. **Compute the Syndrome of the received word** $S(\mathbf{y})$
   - ➜ If there exists $\mathbf{e} \in \mathbb{F}_q^n$ with $w_H(\mathbf{e}) \leq t : S(\mathbf{e}) = S(\mathbf{y}) \Longrightarrow$ **Return:** $\mathbf{y} - \mathbf{e}$
   - ➜ Otherwise, $\Longrightarrow$ **Return:** `FAILURE`

Gilbert-Varshamov bound

GV bound

$$\sum_{i=0}^{d-2}(q-1)^i\binom{n-1}{i} < q^{n-k} \implies \text{Exists an } [n,k,d]_q \text{ code}$$

Proof:

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We construct by induction the columns $h_1, \ldots, h_n \in \mathbb{F}_q^{n-k}$ of H.

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- $h_j \in \mathbb{F}_q^{n-k}$ any vector that is **NOT** a **LC** of $\leq (d-2)$ of $\{h_1, \ldots, h_{j-1}\}$

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Let $j < n$. Exists h_{j+1} with the above property if:

$$\sum_{i=0}^{d-2} \binom{j}{i} (q-1)^i \leq q^{n-k} - 1$$

Number of **LC** of $\leq (d-2)$ of $\{h_1, \ldots, h_j\}$

Total number of vectors in $\mathbb{F}_q^n$ but one

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Gilbert Varshamov distance

Gilbert-Varshamov (GV) distance

The GV distance of an $[n, k]_q$ code is the **maximal integer** d_0 such that:

$$\sum_{i=0}^{d_0-1} \binom{n}{i} (q-1)^i \leq q^{n-k}$$

Number of codewords of a given weight

$$A_w(\mathcal{C}) = |\{\mathbf{c} \in \mathcal{C} \mid w_H(\mathbf{c}) = w\}|$$

Distinct codewords in $\mathcal{C}$ of weight exactly **w**

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In a binary random code: $\mathbb{E}[A_w(\mathcal{C})] = \frac{\binom{n}{w}|\mathcal{C}|}{2^n} = \frac{\binom{n}{w}}{2^{n-k}}$

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In average:

Exists $\mathbf{c} \in \mathcal{C}$ with $\mathrm{w}_H(\mathbf{c}) = w \iff \binom{n}{w} > 2^{n-k}$

$\iff$ w is closed to the GV distance

The Syndrome Decoding (SD) problem

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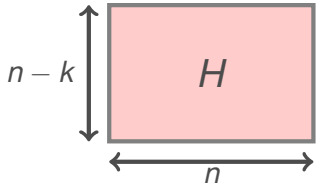

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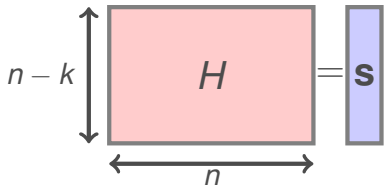

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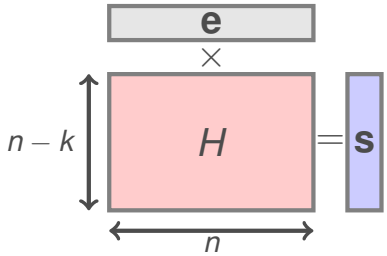

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- A weight $w \in \mathbb{Z}$

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Output

(Decision): Does $\mathbf{e} \in \mathbb{F}_2^n$ of $w_H(\mathbf{e}) \leq w$ such that $\mathbf{e}H^T = \mathbf{s}$ exists? NP-complete

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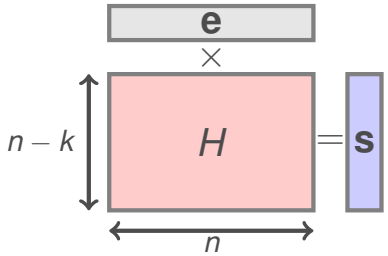

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(Computational): Find $\mathbf{e} \in \mathbb{F}_2^n$ of $w_H(\mathbf{e}) \leq w$ such that $\mathbf{e}H^T = \mathbf{s}$ NP-difficult

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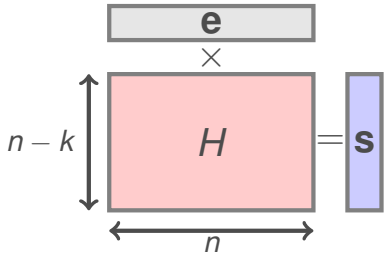

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General Decoding

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- A parity-check matrix $H \in \mathbb{F}_2^{(n-k)\times n}$
  - A received vector $\mathbf{y} \in \mathbb{F}_2^{n-k}$
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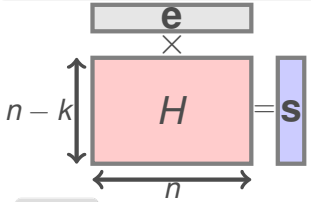

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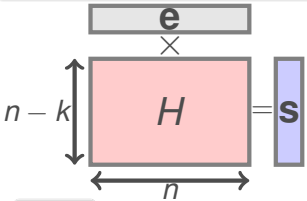
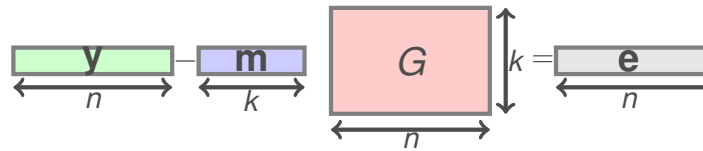
Find $\mathbf{m} \in \mathbb{F}_2^k$ such that $\mathrm{w}_H(\mathbf{y} - m G) \leq w$

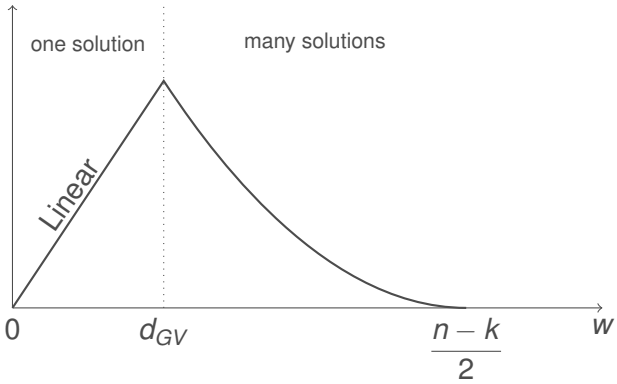




Computational Analysis of Syndrome Decoding

cost (log) of ISD

Binary codes



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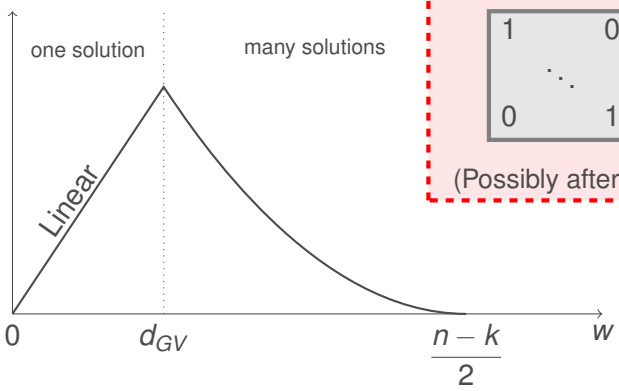

Case $w = \frac{n-k}{2}$

(Possibly after permuting some columns)

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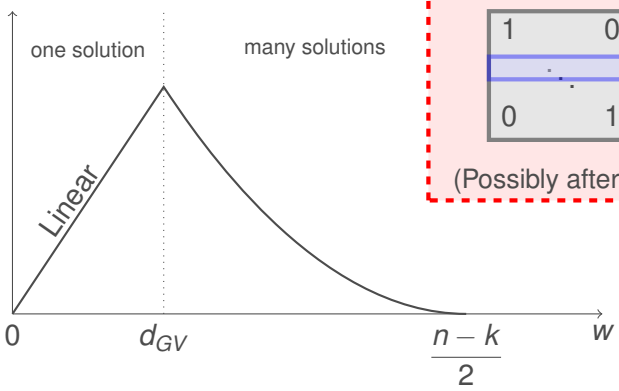

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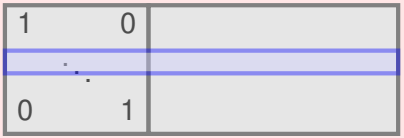

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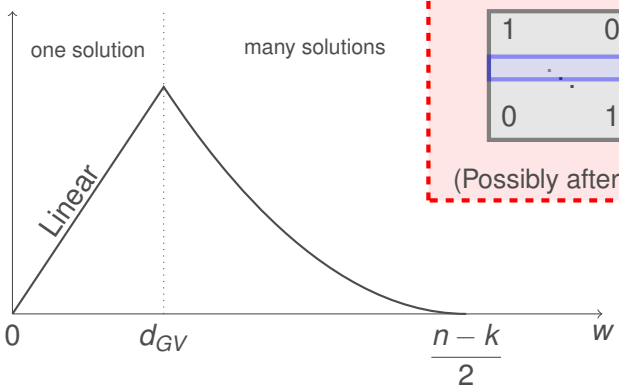
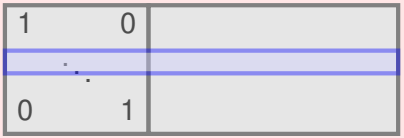


Case $w = \frac{n-k}{2}$



(Possibly after permuting some columns)

In average:

Exists $\mathbf{c} \in \mathcal{C}$ with $\mathrm{w}_H(\mathbf{c}) = w \iff \binom{n}{w} > 2^{n-k}$

$\iff w$ is closed to the GV distance

Trapdoor one-way functions - Decoder

Encoder = Matrix Multiplication

EASY

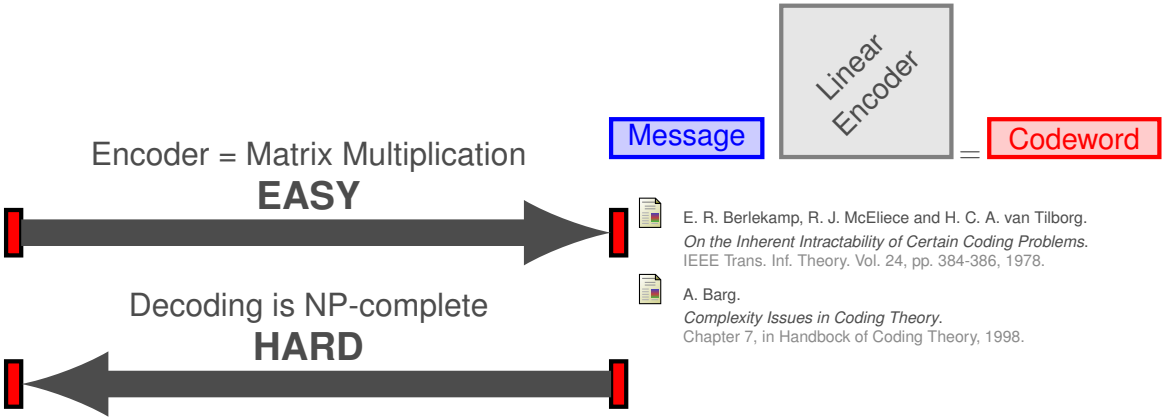
Message

Linear Encoder

$=$

Codeword

Trapdoor one-way functions - Decoder



E. R. Berlekamp, R. J. McEliece and H. C. A. van Tilborg.
On the Inherent Intractability of Certain Coding Problems.
IEEE Trans. Inf. Theory. Vol. 24, pp. 384-386, 1978.

A. Barg.
Complexity Issues in Coding Theory.
Chapter 7, in Handbock of Coding Theory, 1998.

Trapdoor one-way functions - Decoder

Encoder = Matrix Multiplication

EASY

Decoding is NP-complete

HARD

Efficient decoder for certain families of codes

EASY(with TRAPDOOR information)

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1. Error-Correcting Codes and Cryptography

1. Introduction I - Cryptography
2. Introduction II - Coding Theory
3. Encoding (Linear Transformation)
4. Parity Checking
5. Error Correcting Capacity
6. Decoding (A Difficult Problem)
7. **Reed-Solomon Codes**
8. Goppa Codes
9. McEliece Cryptosystem