Code-Based Cryptography

Error-Correcting Codes and Cryptography

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1. Error-Correcting Codes and Cryptography

1. Introduction I - Cryptography
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5. **Error Correcting Capacity**
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Hamming distance

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$

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The **Hamming weight** of $\mathbf{x}$ is $\mathrm{w}_H(\mathbf{x}) = \{i \mid x_i \neq 0\}$

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Example

$$\begin{array}{l} \mathbf{x} \to 1\ 1\ 1\ 1 \\ \hline \mathbf{y} \to 1\ 0\ 1\ 0 \end{array} \longrightarrow d_H(\mathbf{x}, \mathbf{y}) = 2$$

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Example

$$\begin{array}{l}
\mathbf{x} \rightarrow \ \mathrm{I\,U\,T} \\
\hline
\mathbf{y} \rightarrow \ \mathrm{D\,U\,T}
\end{array} \longrightarrow d_H(\mathbf{x}, \mathbf{y}) = 1$$

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Example

$$\mathbf{x} \to \underset{}{2}\ \underset{\uparrow}{0}\ \underset{\uparrow}{0}\ \underset{}{1}\ \underset{\uparrow}{0} \longrightarrow \mathrm{w}_H(\mathbf{x}) = 2$$

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The Hamming distance is a metric on $\mathbb{F}_q^n$

→ With this distance $\mathbb{F}_q^n$ becomes a metric space.

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For all $\mathbf{x},\ \mathbf{y},\ \mathbf{z} \in \mathbb{F}_q^n$, the following conditions hold:

- NON-NEGATIVITY: $d_H(\mathbf{x}, \mathbf{y}) \geq 0$.

  Moreover, $d_H(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$.

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- TRIANGLE INEQUALITY: $d_H(\mathbf{x}, \mathbf{y}) \leq d_H(\mathbf{x}, \mathbf{z}) + d_H(\mathbf{z}, \mathbf{y})$

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Minimum distance

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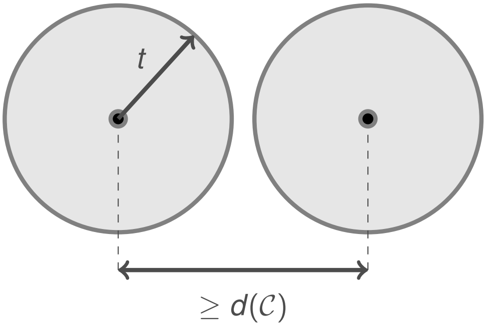
The **minimum distance** of $\mathcal{C}$ is $d(\mathcal{C}) = \min\{d_H(\mathbf{c}_1, \mathbf{c}_2) \mid \mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C} \text{ and } \mathbf{c}_1 \neq \mathbf{c}_2\}$



Minimum distance = Minimum weight

Proposition 1

Let $\mathcal{C}$ be a linear code.

$$\boxed{\mathrm{w}(\mathcal{C}) = d(\mathcal{C})}$$

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Proof:

Since $\mathcal{C}$ is a linear code we have that:

$$\mathbf{0} \in \mathcal{C} \quad \text{and} \quad \mathbf{c}_1 - \mathbf{c}_2 \in \mathcal{C}, \forall \mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}$$

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Then, the results follows from the fact that:

$$\mathrm{w}_H(\mathbf{c}) = d_H(\mathbf{0}, \mathbf{c}) \quad \text{and} \quad d_H(\mathbf{c}_1, \mathbf{c}_2) = \mathrm{w}_H(\mathbf{c}_1 - \mathbf{c}_2)$$

Minimum distance - Parity check matrix

Proposition 2:

Let $\mathcal{C}$ be an $[n,k]_q$ code with parity check matrix H:

$$d(\mathcal{C}) = d \iff \text{Every set of } (d-1) \text{ columns of } H \text{ are linearly independent}$$

Proof:

Let $H \in \mathbb{F}_q^{(n-k)\times n}$ be a parity check matrix for $\mathcal{C}$.
It is easy to check that:

$$\exists \mathbf{c} \in \mathcal{C}, \mathbf{c} \neq \mathbf{0} \;:\; \mathrm{w}_H(\mathbf{c}) = w \iff \exists w \text{ columns of } H \text{ Linearly dependent}$$

Moreover, since $d(\mathcal{C}) = \mathrm{w}(\mathcal{C})$, then the weight d is achieved by some codeword. That is,

$$\exists \mathbf{c} \in \mathcal{C} \;:\; \mathrm{w}_H(\mathbf{c}) = d$$

Or equivalently, d is the minimal number of columns required for linear dependence.

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3. <u>In the binary case:</u>

$$d(\mathcal{C}) = 2 \iff H \text{ has two columns } h_i, h_j \ : \ h_i = h_j$$

$$d(\mathcal{C}) = 3 \iff \begin{cases} H \text{ has no zero columns} \\ \text{All columns are mutually distinct} \end{cases}$$

Singleton Bound

Proposition 3: Singleton bound

Let $\mathcal{C}$ be an $[n, k]_q$ code. Then $d(\mathcal{C}) \leq n - k + 1$

This bound is the **SINGLETON BOUND**

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Proof:

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- ➜ The **rank** of a parity check matrix H for $\mathcal{C}$ is $n - k$.
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Proof:

- ➜ The **rank** of a parity check matrix H for $\mathcal{C}$ is $n - k$.
- ➜ At most $n - k + 1$ columns of H are **linearly dependent**
- ➜ By Proposition 2: $d(\mathcal{C}) \leq n - k + 1$

Error-detecting & Error-correcting capability

$\mathbf{c} \in \mathcal{C}$
is sent

Noisy

$\mathbf{y} = \mathbf{c} + \mathbf{e} \in \mathbb{F}_q^n$
is received

$\mathbf{e} \in \mathbb{F}_q^n$ is called
Error Pattern

Example

Code	Length	Up	Down	Left	Right

Note that:

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$\mathcal{C}_1$	2	00	10	01	11

Note that:

- $\mathcal{C}_1$ can not detect errors.

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Code	Length	Up	Down	Left	Right
$\mathcal{C}_1$	2	00	10	01	11
$\mathcal{C}_2$	3	000	110	011	101

Note that:

- $\mathcal{C}_1$ can not detect errors.
- $\mathcal{C}_2$ can detect but not correct 1 error.

Example

Code	Length	Up	Down	Left	Right
$\mathcal{C}_1$	2	00	10	01	11
$\mathcal{C}_2$	3	000	110	011	101
$\mathcal{C}_3$	6	000000	111000	001110	110011

Note that:

- $\mathcal{C}_1$ can not detect errors.
- $\mathcal{C}_2$ can detect but not correct 1 error.
- $\mathcal{C}_3$ can detect and correct up to 1 error.

Error-detection capability

Detectable errors

Let $\mathcal{C}$ be an $[n, k]_q$ code of minimum distance d:

Any error pattern of size at most $d-1$ can be **detected**.

Let $\mathbf{c} \in \mathcal{C}$ be the transmitted codeword and $\mathbf{e}$ be the error pattern.

Take notice that: If $\mathrm{w}_H(\mathbf{e}) \leq d-1 \implies \mathbf{y} = \mathbf{c} + \mathbf{e} \notin \mathcal{C}$

Error-detection capability

Let $\mathbf{c} \in \mathcal{C}$ be the transmitted codeword and $\mathbf{e}$ be the error pattern:

➜ Some error patterns $\mathbf{e} \in \mathbb{F}_q^n \ : \ \mathrm{w}_H(\mathbf{e}) \geq d$ can be **detected**

➜ **Error detection** fails when $\mathbf{e} \in \mathcal{C}$ and $\mathbf{e} \neq 0$.

Error-detection capability

Let $\mathbf{c} \in \mathcal{C}$ be the transmitted codeword and $\mathbf{e}$ be the error pattern:

- ➜ Some error patterns $\mathbf{e} \in \mathbb{F}_q^n \;:\; \mathrm{w}_H(\mathbf{e}) \geq d$ can be **detected**

- ➜ **Error detection** fails when $\mathbf{e} \in \mathcal{C}$ and $\mathbf{e} \neq 0$.

Number of detectable errors:

Let $\mathcal{C}$ be an $[n, k]_q$ code.

There are $q^n - q^k$ error patterns that can be **detected**.

Error-correcting capability

$\mathbf{c} \in \mathcal{C}$ is sent $\longrightarrow$ $\mathbf{y} \in \mathbb{F}_q^n$ is received

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What strategy?

Error-correcting capability

$\mathbf{c} \in \mathcal{C}$ is sent → $\mathbf{y} \in \mathbb{F}_q^n$ is received

What strategy?

Look for $\mathbf{x} \in \mathcal{C}$: $d_H(\mathbf{y}, \mathbf{x})$ is minimized
(Minimum Distance Decoding)

Error-correcting capability

Theorem:

Let $\mathcal{C}$ be a linear code with minimum distance d:

$$\mathcal{C} \text{ can correct } t \text{ errors} \quad \Longleftrightarrow \quad t < \frac{d}{2}, \text{ i.e. } t \leq \left\lfloor \frac{d-1}{2} \right\rfloor$$

Proof:

Let $\mathbf{y}$ be the received word and suppose that t errors have occurred.
If $\mathcal{C}$ cannot correct this error pattern then there are two codewords at distance t from the received codeword.

$$\left. \begin{array}{l} \exists \mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C} \\ \exists \mathbf{e}_1, \mathbf{e}_2 \in \mathbb{F}_q \text{ with } \mathrm{w}_H(\mathbf{e}_1), \mathrm{w}_H(\mathbf{e}_2) < t \end{array} \right\} \text{ such that } \mathbf{y} = \mathbf{c}_1 + \mathbf{e}_1 = \mathbf{c}_2 + \mathbf{e}_2$$

Thus we have a nonzero codeword of weight smaller than d, i.e.

$$\mathrm{w}_H(\mathbf{c}_1 - \mathbf{c}_2) = \mathrm{w}_H(\mathbf{e}_1 - \mathbf{e}_2) < 2t < d$$

which contradicts the minimality of d.

The rest is left as an exercise.

Error-detecting & Error-correcting capability

Minimum distance $d(\mathcal{C})$ determines capabilities of the code $\mathcal{C}$

- **Number of detectable errors:** $\hat{t} = d(\mathcal{C}) - 1$
- **Number of correctable errors:** $t = \left\lfloor \frac{d(\mathcal{C})-1}{2} \right\rfloor$

Objectives of Coding Theory

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- ➜ The complexity of the **encoding** and **decoding** procedures

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➜ **The Information Rate:** $\frac{k}{n}$

➜ **The relative minimum distance** $\frac{d}{n}$

➜ The complexity of the **encoding** and **decoding** procedures

The goal of **Coding Theory** is to provide codes with:

- ↗ High information rate
- ↗ High error-correction rate
- ↘ Low complexity of encoding and decoding

1. Error-Correcting Codes and Cryptography