Code-Based Cryptography

Error-Correcting Codes and Cryptography

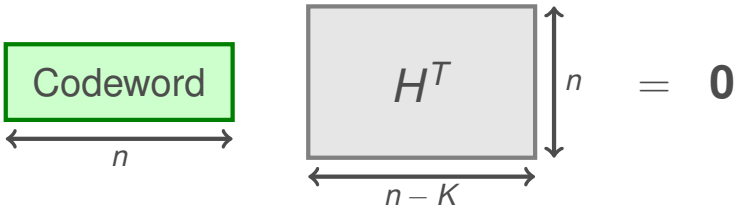
I. Márquez-Corbella

informatiques mathématiques

1. Error-Correcting Codes and Cryptography

1. Introduction I - Cryptography
2. Introduction II - Coding Theory
3. Encoding (Linear Transformation)
4. **Parity Checking**
5. Error Correcting Capacity
6. Decoding (A Difficult Problem)
7. Reed-Solomon Codes
8. Goppa Codes
9. McEliece Cryptosystem

Parity Check Matrix



Parity check matrix

Let $\mathcal{C}$ be an $[n, k]_q$ code.

H is a **parity check matrix** of $\mathcal{C}$ $\iff$ $\mathcal{C}$ is the **null space** of H

That is:

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{F}_q^n \mid \mathbf{x}H^T = 0\}$$

[7,4]-Hamming code

$$\underbrace{(m_1, m_2, m_3, m_4)}_{\text{Information bits}}$$

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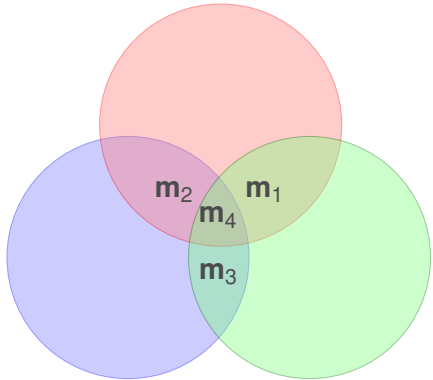

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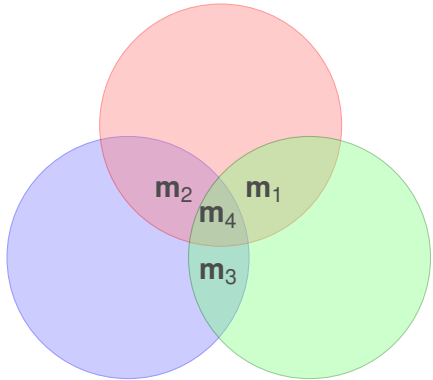

The number of ones
in every circle is even

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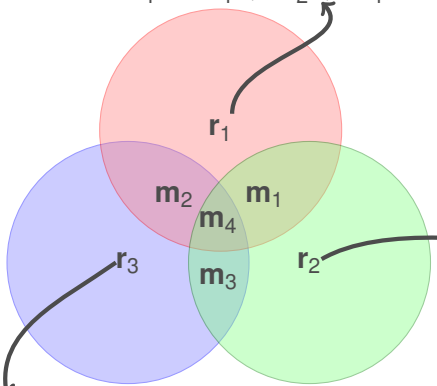
$\mathbf{r}_1 = \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_4 \mod 2$



The number of ones in every circle is even

$\mathbf{r}_2 = \mathbf{m}_1 + \mathbf{m}_3 + \mathbf{m}_4 \mod 2$

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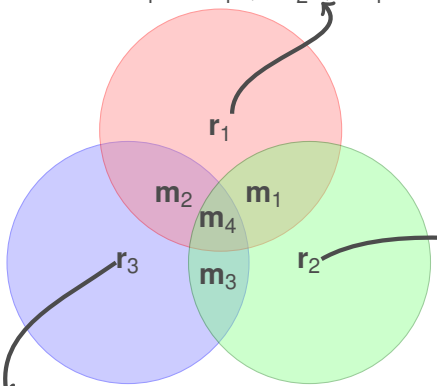

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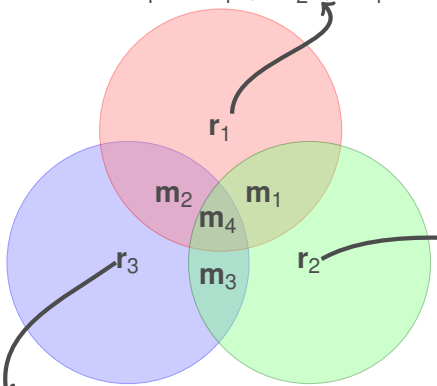

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[7,4]-Hamming code

The **redundant information** can be obtained from the message by 3 **parity checks**:

$$\begin{cases} \mathbf{r}_1 &= \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_4 \\ \mathbf{r}_2 &= \mathbf{m}_1 + \mathbf{m}_3 + \mathbf{m}_4 \\ \mathbf{r}_3 &= \mathbf{m}_2 + \mathbf{m}_3 + \mathbf{m}_4 \end{cases}$$

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$\mathbf{c} = (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_4)$ is a codeword $\iff$ $H\mathbf{c}^T = 0$

with $H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{F}_2^{3\times 7}$

Binary Hamming Codes

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$H \in \mathbb{F}_2^{r \times 2^r - 1}$ such that H contains **all** nonzero binary r-tuples **exactly once** as a column

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Binary Hamming codes correct up to 1 error.

Parity Check Matrix

A code can have more than one parity-check matrix!

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Proposition: Characterization of a parity-check matrix

Let $\mathcal{C}$ be an $[n, k]_q$ code with generator matrix G

$$H \text{ is a parity check-matrix of } \mathcal{C} \iff GH^T = 0$$

Proof:

From the definition of parity check matrix: $\mathbf{c}H^T = \mathbf{0}$, for all $\mathbf{c} \in \mathcal{C}$

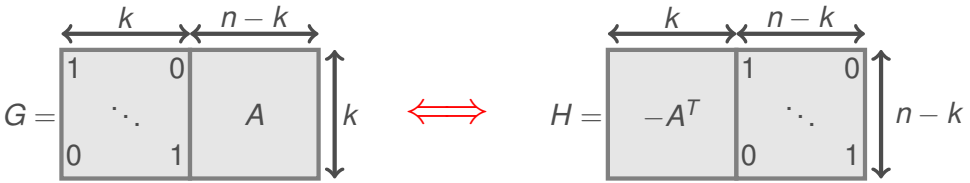
Recall that every codeword is of the form: $\mathbf{c} = \mathbf{m}G$ with $\mathbf{m} \in \mathbb{F}_q^k$

Thus, $(\mathbf{m}G)\, H^T = \mathbf{0}$, for all $\mathbf{m} \in \mathbb{F}_q^k$

And we conclude that $GH^T = 0$

Parity Check Matrix

Proposition: How to get a parity check-matrix?



is a generator matrix for $\mathcal{C}$ ⟺ is a parity-check matrix for $\mathcal{C}$

Proof:

"$\Longrightarrow$" We clearly have $HG^T = 0 = -A^T + A^T$
Thus, $\mathcal{C} \subseteq \ker(H)$
Since $\mathrm{rank}(H) = n - k \implies \dim(\ker(H)) = k = \dim(\mathcal{C})$
Hence, H is a parity check matrix for $\mathcal{C}$

"$\Longleftarrow$" The converse is proved similarly.

Dual Code

G is a generator matrix for $\mathcal{C}$ $\iff$ H is a generator matrix for $\mathcal{C}^{\perp}$

The dual code

Let $\mathcal{C}$ be an $[n, k]_q$ code. We define the **dual code** $\mathcal{C}^{\perp}$ as

$$\mathcal{C}^{\perp} = \{\mathbf{x} \in \mathbf{F}_q^n \mid \mathbf{x} \cdot \quad \mathbf{c} = 0\ ,\ \forall \mathbf{c} \in \mathcal{C}\}$$

Dual Code

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The dual code

Let $\mathcal{C}$ be an $[n,k]_q$ code. We define the **dual code** $\mathcal{C}^{\perp}$ as

$$\mathcal{C}^{\perp} = \{\mathbf{x} \in \mathbf{F}_q^n \mid \mathbf{x} \cdot \mathbf{c} = 0\,,\ \forall \mathbf{c} \in \mathcal{C}\}$$

For $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$ the **inner product** is defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Dual Code

Proposition:

Let $\mathcal{C}$ be an $[n, k]_q$ code. Then,

the **dual code** $\mathcal{C}^\perp$ is an $[n, n-k]_q$ **code**.

Proof:

From the definition of dual code, the following statements are equivalents:

$$\begin{aligned} \mathbf{x} \in \mathcal{C}^\perp &\iff \mathbf{c} \cdot \mathbf{x} = \mathbf{0} \text{ , for all } \mathbf{c} \in \mathcal{C} \\ &\iff \mathbf{m}G\mathbf{x}^T = \mathbf{0} \text{ , for all } \mathbf{m} \in \mathbb{F}_q^k \\ &\iff G\mathbf{x}^T = 0 \end{aligned}$$

Thus, $\mathcal{C}^\perp = \ker(G)$

Moreover, since $\operatorname{rank}(G) = k \implies \dim(\mathcal{C}^\perp) = n - k$

We can also deduce that G is a parity check matrix for $\mathcal{C}^\perp$.

Dual Code

Proposition:

Let $\mathcal{C}$ be an $[n, k]_q$ code with generator matrix G. Then,

$$(\mathcal{C}^{\perp})^{\perp} = \mathcal{C}$$

Trapdoor one-way functions - Decoder

Encoder = Matrix Multiplication

EASY

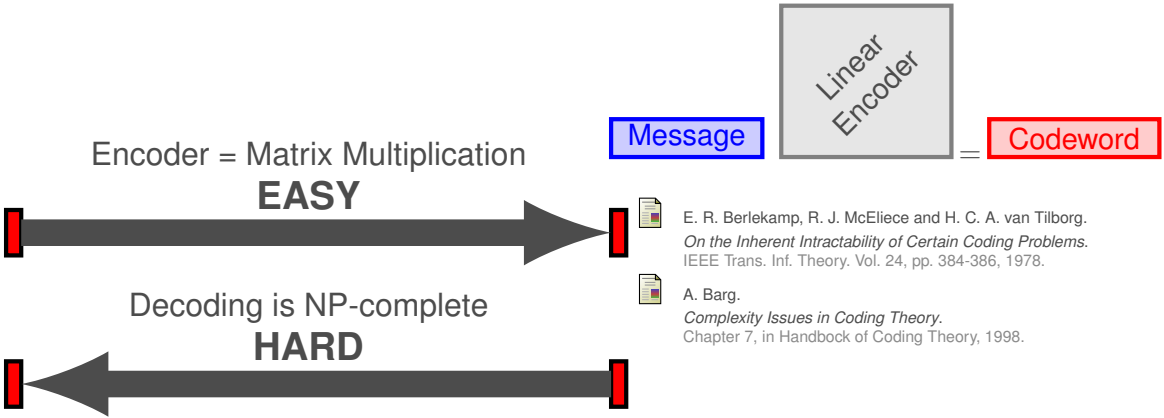
Message

Linear Encoder

$=$

Codeword

Trapdoor one-way functions - Decoder



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Complexity Issues in Coding Theory.
Chapter 7, in Handbock of Coding Theory, 1998.

Trapdoor one-way functions - Decoder

Encoder = Matrix Multiplication

EASY

Decoding is NP-complete

HARD

Efficient decoder for certain families of codes

EASY(with TRAPDOOR information)

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